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LARGE DEFORMATION POSSIBLE IN EVERY ISOTROPIC ELASTIC MEMBRANE

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CONTENTS

This paper is concerned with static solutions of finitely deformed elastic membranes regarded as thin shells. It deals with deformations that can be maintained, in the absence of body force, in every isotropic elastic membrane by the application of edge loads and/or uniform normal surface loads on the major surfaces of the thin shell-like body. The solutions, which are valid for both compressible and incompressible materials, are obtained with the use of a strain energy response function which depends on the metric tensor of the membrane in its deformed configuration. The main results are summarized by several theorems and their corollaries in accordance with three mutually exclusive cases for which the initial undeformed surface of the membrane (which may be a sector of a complete or closed surface) is, respectively, developable, spherical and a surface of variable Gaussian curvature satisfying certain differential criteria. The corresponding deformed surfaces are, respectively, a plane or a right circular cylinder, a sphere and a surface of constant mean curvature. These results are exhaustive in that they represent all finite deformation solutions possible in every isotropic elastic material characterized by the strain energy response mentioned above. Also discussed are some special cases of the general results and several families of solutions in terms of an alternative description which should be useful in application and which permit easy interpretations.

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1. Introduction

In the three-dimensional theory of finite elasticity a state of deformation possible in every isotropic material, in the absence of body force, is often referred to in the current literature as a controllable (or a universal) state. A programme of fruitful research for such solutions in the case of homogeneous, isotropic and incompressible elastic material was initiated by Rivlin nearly three decades ago. (References to Rivlin's original contributions and others on the subject can be found in Green & Adkins (1970) and Truesdell & Noll (1965).) Subsequently, Ericksen (1954) considered the problem of finding all of the solutions possible in every isotropic incompressible material but the class of deformations found by him was not exhaustive. In recent years a number of additional solutions have been found by others but the question of completeness of solutions found remains unresolved in the context of the three-dimensional theory. It should be remarked here that an important application of the finite deformation solutions referred to above arises from the fact that they permit direct determination of elastic response functions from comparisons of experiments and predictions of the theory. Indeed, such a procedure was used by Rivlin & Saunders (1951) for determining the form of the constitutive relations of various rubbers regarded as incompressible elastic materials.

By way of additional background, it should be mentioned that some controllable solutions for finitely deformed elastic membranes have been discussed previously by Adkins & Rivlin (1952), which include a detailed study of the inflation of a circular plane sheet and the inflation of a closed spherical membrane. In the latter case, the formula for the internal pressure required to inflate the membrane recovers more directly the results of a solution of Green & Shield (1950) for thick spherical shells when these are specialized to *thin* shells by assuming that the ratio of thickness to radius is much smaller than unity. In addition, a class of controllable solutions for axisymmetrically deformed membranes of revolution is given by Green & Adkins (1970, ch. 4).

The present paper is concerned with static solutions of finitely deformed elastic membranes regarded as *thin* shells. It deals with deformations that can be maintained, in the absence of body force, in every isotropic elastic material by the application of edge loads and/or uniform normal surface loads on the major surfaces of the thin shell-like body. As in the three-dimensional theory, for brevity we adopt the terminology of *controllable* states or solutions also in the present development.‡ The controllable solutions, which are valid for both compressible and incompressible materials, are obtained with the use of a strain energy response function which depends explicitly on the metric tensor of the membrane in its deformed configuration, or equivalently on the coefficients of the first fundamental form of the deformed surfaces.§ Alternatively the strain energy may be expressed in terms of a relative strain measure, i.e., in terms of the difference of the metric tensor of the deformed state and that of the initial undeformed state, but this form is not employed here.

The main results derived are summarized in § 6 by three theorems, namely theorems 6.5–6.7, and several corollaries in accordance with three mutually exclusive cases for which the initial undeformed surface (which may be a sector of a complete or closed surface) is, respectively, developable, spherical and a surface of variable Gaussian curvature satisfying a certain differ-

‡ Here again this terminology refers to a class of deformation if it can be maintained in every material of that class in the absence of body force.

§ In the present development the strain energy is assumed not to depend explicitly on the reference geometry, i.e. on the coefficients of the second fundamental form in the reference state.

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ential criterion. The corresponding deformed surfaces must be, respectively, a plane or a right circular cylinder, a sphere and a surface of constant mean curvature. In all cases the controllable deformation results in a homogeneous strain on the deformed surface. Moreover, in the case of an initial surface with a variable Gaussian curvature, the deformation results in a conformal correspondence between the initial and the deformed surface, while in the case of an initially spherical surface a net of orthogonal trajectories on the initial surface must be deformed into a net of lines of curvature on the deformed surface. It should be remarked that the solutions obtained are both necessary and sufficient for the satisfaction of the system of differential equations resulting from compatibility and equilibrium in the absence of body force, after using also the relevant constitutive equations. The results obtained are therefore exhaustive, i.e. for the class of elastic material characterized by the strain energy response mentioned above, we have obtained here all possible controllable solutions.

The differential criterion referred to above essentially imposes a restriction on the variable Gaussian curvature of the initial undeformed surface in order to ensure the existence of the second fundamental form of the deformed surface (see the equation (5.22)). In general, it is extremely difficult to integrate this second order partial differential equation and obtain more explicit information regarding the shape of the initial surface. However, whenever the differential criterion is satisfied, our solutions in the deformed configuration can be expressed explicitly in terms of the initial geometry of the undeformed membrane.

After collecting in § 2 various formulae from the theory of a surface embedded in a Euclidean 3-space and also summarizing the main results from the nonlinear theory of elastic membranes by direct approach, in § 3 we show equivalence between the development obtained by direct approach and the corresponding results derived from the three-dimensional equations of nonlinear elasticity as given by Green & Adkins (1970) for both compressible and incompressible materials. Next, the problem of controllable deformation of an isotropic, elastic membrane is formulated in § 4 and a summary of the solutions of the first and the second fundamental forms of the deformed surface are given in § 5 with details of their derivations outlined in two appendices A and B. By using the latter results, static controllable solutions are finally obtained in § 6, which lead to the theorems 6.5–6.7 mentioned earlier.

The differential criterion mentioned above is examined in the case of an initial surface with a variable Gaussian curvature for surfaces of constant mean curvatures and negative Gaussian curvatures; the results are summarized as theorem 6.4 and corollaries 6.1, 6.2 and 6.3. One of these, namely corollary 6.1, states that in the absence of the surface load, among all non-developable surfaces of constant mean curvature only a nondevelopable minimal surface is a possible initial surface in a controllable deformation. Corollary 6.3 shows that a minimal surface can only be controllably deformed into another minimal surface, provided that the surface load is absent. When a surface of constant mean curvature (and also of negative Gaussian curvature) is controllably deformed into another surface of constant mean curvature under a nonzero uniform normal surface load, the mean curvatures of the initial and the deformed surfaces are proportional to each other (corollary 6.2).

The main results of § 6 are next employed in § 7 which deals with certain classes of membranes of variable Gaussian curvatures in the initial undeformed state when at least either the deformed or the initial undeformed configuration is in the form of a surface of revolution. In particular, the development of § 7 leads to the corollaries 7.1 and 7.2 concerning the character of the deformed surface when the corresponding initial surface is a sector of nondevelopable surface of

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revolution. With the help of corollaries 7.1 and 7.2, we also obtain controllable deformations for an elastic membrane in which both the initial and the deformed surfaces are complete surfaces of revolution. These results are in complete agreement with those obtained by Green & Adkins (1970, §§ 11 and 14) from the three-dimensional formulation. The corollaries 7.3 and 7.4 are appropriate for the case in which the deformed surface is a sector of a surface of revolution. One of these, namely corollary 7.3, states that the only possible nondevelopable surface of revolution resulting from a controllable deformation under the action of edge loads alone is a catenary. Finally, in § 8, we consider some special cases of the general results of § 6. Employing an alternative description for the deformation of the initial undeformed surface into the deformed surface of the membrane, we discuss five particular families of controllable deformations in which each of the initial and the deformed surfaces may be a plane, a right circular cylinder or a sphere. The corresponding solutions of these five families permit easy interpretations of the general results and should be useful in application.

2. Notation and basic equations

We are concerned here with static deformation of a two-dimensional material surface, embedded in a Euclidean three-space $\mathscr{E}^3$. Let convected coordinates θ^α $(\alpha = 1, 2)$ be assigned to each particle (or material point) of the surface, let the surface occupied by the material surface in the deformed and the undeformed reference configurations be referred to as s and $\mathscr{S}$, respectively, and let $\boldsymbol{R}$ denote the position vector, relative to a fixed origin, of a typical point of $\mathscr{S}$. Then, $\boldsymbol{R} = \boldsymbol{R}(\theta^\alpha)$ specifies the place occupied by the material point in the undeformed configuration which we take to be the initial configuration. Likewise, the position vector of s, relative to the same fixed origin, is given by $\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha)$ and this specifies the place occupied by the material point θ^α in the deformed configuration.

Let $\boldsymbol{a}_\alpha$ denote the base vectors along the θ^α-curves on s and let $\boldsymbol{a}_3$ be the unit normal to s. Then,

$$\left.\begin{aligned} \boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}, \quad \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = a_{\alpha\beta}, \quad a = \det(a_{\alpha\beta}) > 0, \\ \boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta, \quad \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta = a^{\alpha\beta}, \quad a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta, \quad \boldsymbol{a}^\alpha = a^{\alpha\gamma}\boldsymbol{a}_\gamma \end{aligned}\right\} \tag{2.1}$$

and

$$\left.\begin{aligned} \boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta = \epsilon_{\alpha\beta}\boldsymbol{a}^3, \\ \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_3 = 0, \quad \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 1, \quad \boldsymbol{a}^3 = \boldsymbol{a}_3, \end{aligned}\right\} \tag{2.2}$$

where a comma stands for partial differentiation with respect to θ^α, $a_{\alpha\beta}$ and $a^{\alpha\beta}$ are the components of the first fundamental form of s and its conjugate, δ^α_β is the Kronecker symbol in 2-space, and $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$ are the ϵ-systems for the surface s defined by

$$\left.\begin{aligned} \epsilon_{\alpha\beta} = a^{\frac{1}{2}} e_{\alpha\beta}, \quad \epsilon^{\alpha\beta} = a^{-\frac{1}{2}} e^{\alpha\beta}, \quad \epsilon_{\alpha\gamma}\,\epsilon^{\beta\gamma} = \delta^\beta_\alpha, \\ e_{11} = e_{22} = e^{11} = e^{22} = 0, \quad e_{12} = e^{12} = -e_{21} = -e^{21} = 1. \end{aligned}\right\} \tag{2.3}$$

Also, we note the relations $\quad a_{\alpha\beta} = \epsilon_{\alpha\xi}\,\epsilon_{\beta\eta}\, a^{\xi\eta}, \quad a^{\alpha\beta} = \epsilon^{\alpha\xi}\epsilon^{\beta\eta} a_{\xi\eta},$ (2.4)

as well as the identities

$$\epsilon_{\alpha\xi}\,\epsilon_{\beta\eta} = a_{\alpha\beta}a_{\xi\eta} - a_{\alpha\eta}a_{\beta\xi}, \quad \epsilon^{\alpha\xi}\epsilon^{\beta\eta} = a^{\alpha\beta}a^{\xi\eta} - a^{\alpha\eta}a^{\beta\xi}. \tag{2.5}$$

For later reference, we recall the formulae

$$\left.\begin{aligned} T_{\alpha\beta|\gamma} &= T_{\alpha\beta,\gamma} - \Gamma^\sigma_{\alpha\gamma}T_{\sigma\beta} - \Gamma^\sigma_{\beta\gamma}T_{\alpha\sigma}, \\ T^{\alpha\beta}{}_{|\gamma} &= T^{\alpha\beta}{}_{,\gamma} + \Gamma^\alpha_{\gamma\sigma}T^{\sigma\beta} + \Gamma^\beta_{\gamma\sigma}T^{\alpha\sigma}, \\ T_{\alpha|\beta\gamma} - T_{\alpha|\gamma\beta} &= R^{\lambda}{}_{\cdot\alpha\beta\gamma}T_\lambda, \\ R^{\lambda}{}_{\cdot\alpha\beta\gamma} &= \Gamma^\lambda_{\alpha\gamma,\beta} - \Gamma^\lambda_{\alpha\beta,\gamma} + \Gamma^\mu_{\alpha\gamma}\Gamma^\lambda_{\mu\beta} - \Gamma^\mu_{\alpha\beta}\Gamma^\lambda_{\mu\gamma}, \\ R_{\lambda\alpha\beta\gamma} &= a_{\lambda\sigma}R^{\sigma}{}_{\cdot\alpha\beta\gamma}, \quad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}, \end{aligned}\right\} \tag{2.6}$$

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where a vertical bar (|) stands for covariant differentiation with respect to the metric tensor $a_{\alpha\beta}$, $R^{\lambda}_{\cdot\alpha\beta\gamma}$ and $R_{\lambda\alpha\beta\gamma}$ are the curvature tensors for the surface s and the surface Christoffel symbols in (2.6) are defined by

$$\left.\begin{aligned}\Gamma^{\sigma}_{\alpha\beta} &= a^{\sigma\gamma}\Gamma_{\alpha\beta\gamma},\\ \Gamma_{\alpha\beta\gamma} &= a_{\gamma\sigma}\Gamma^{\sigma}_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\gamma,\beta}+a_{\beta\gamma,\alpha}-a_{\alpha\beta,\gamma}).\end{aligned}\right\} \tag{2.7}$$

We record here the expressions for the partial derivatives of $a_{\alpha\beta}$, $a^{\alpha\beta}$ and the determinant a which can be expressed in terms of the Christoffel symbols. Thus

$$\left.\begin{aligned}a_{\alpha\beta,\gamma} &= a_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma}+a_{\beta\sigma}\Gamma^{\sigma}_{\alpha\gamma} = \Gamma_{\alpha\gamma\beta}+\Gamma_{\beta\gamma\alpha},\\ a^{\alpha\beta}{}_{,\gamma} &= -a^{\alpha\sigma}\Gamma^{\beta}_{\sigma\gamma}-a^{\beta\sigma}\Gamma^{\alpha}_{\sigma\gamma},\\ (a^{\frac{1}{2}})_{,\gamma} &= a^{\frac{1}{2}}\Gamma^{\sigma}_{\sigma\gamma}\end{aligned}\right\} \tag{2.8}$$

and we note that

$$a_{\alpha\beta|\gamma} = a^{\alpha\beta}{}_{|\gamma} = 0, \quad \epsilon_{\alpha\beta|\gamma} = \epsilon^{\alpha\beta}{}_{|\gamma} = 0. \tag{2.9}$$

In the above equations and throughout the paper, all Greek indices take the values 1, 2, Latin indices take the values 1, 2, 3 and the usual summation convention over a repeated index (one subscript and one superscript) is employed. Also, the raising and the lowering of indices of the components of surface tensors is accomplished with the use of $a_{\alpha\beta}$ and $a^{\alpha\beta}$. Whenever possible, in what follows, we use capital letters to represent the duals of quantities associated with s in the reference surface $\mathscr{S}$. For example, the base vectors and the components of the metric tensor of the reference surface $\mathscr{S}$ will be designated by $\mathbf{A}_i$ and by $A_{\alpha\beta}$, $A^{\alpha\beta}$. On the other hand, for such quantities as the Christoffel symbols, the ϵ-systems and the curvature tensor associated with the surface $\mathscr{S}$, we use the same letters to which we add an overbar. Generally, we need only to employ covariant derivatives with respect to the metric tensor of the surface s; however, if it becomes necessary to exhibit covariant differentiation with respect to $A_{\alpha\beta}$, we indicate this with the use of a dagger (†). We note that results similar to those in (2.1)–(2.9) hold also for the surface $\mathscr{S}$. In particular, we record the following formulae:

$$\bar{\epsilon}_{\alpha\beta} = A^{\frac{1}{2}}e_{\alpha\beta}, \quad \bar{\epsilon}^{\alpha\beta} = A^{-\frac{1}{2}}e^{\alpha\beta}, \quad A = \det(A_{\alpha\beta}) > 0, \tag{2.10}$$

$$\bar{\Gamma}^{\sigma}_{\alpha\beta} = A^{\sigma\gamma}\bar{\Gamma}_{\alpha\beta\gamma}, \quad \bar{\Gamma}_{\alpha\beta\gamma} = A_{\gamma\sigma}\bar{\Gamma}^{\sigma}_{\alpha\beta} = \tfrac{1}{2}(A_{\alpha\gamma,\beta}+A_{\beta\gamma,\alpha}-A_{\alpha\beta,\gamma}), \quad \bar{\Gamma}_{\beta\gamma\lambda}\bar{\Gamma}^{\lambda}_{\mu\delta} = \bar{\Gamma}^{\lambda}_{\beta\gamma}\bar{\Gamma}_{\mu\delta\lambda},$$

and

$$\bar{R}_{\lambda\alpha\beta\gamma} = \bar{\Gamma}_{\alpha\gamma\lambda,\beta}-\bar{\Gamma}_{\alpha\beta\lambda,\gamma}+\bar{\Gamma}_{\lambda\gamma\mu}\bar{\Gamma}^{\mu}_{\alpha\beta}-\bar{\Gamma}_{\lambda\beta\mu}\bar{\Gamma}^{\mu}_{\alpha\gamma}, \tag{2.11}$$

which can also be expressed in terms of $A_{\alpha\beta}$ and its partial derivatives.

For future use, we express now the ratio a/A and $a^{\alpha\beta}$ in terms of certain invariants associated with the metric tensor of s. Put

$$c^{\alpha}_{\gamma} = A^{\alpha\xi}a_{\xi\gamma} \tag{2.12}$$

and introduce the invariants I_1 and I_2 by

$$I_1 = c^{\gamma}_{\gamma} = A^{\alpha\beta}a_{\alpha\beta}, \quad I_2 = c^{\alpha}_{\gamma}c^{\gamma}_{\alpha} = A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}a_{\beta\alpha}. \tag{2.13}$$

By the identity

$$2\det(c^{\alpha}_{\gamma}) = 2(c^1_1c^2_2-c^1_2c^2_1) = (c^{\gamma}_{\gamma})^2-c^{\alpha}_{\eta}c^{\eta}_{\alpha} = I_1^2-I_2$$

and the expression for the determinant of (2.12), namely

$$\det(c^{\alpha}_{\gamma}) = \det(a_{\xi\gamma})[\det(A_{\alpha\sigma})]^{-1} = a/A,$$

we arrive at

$$a/A = \tfrac{1}{2}(I_1^2-I_2). \tag{2.14}$$

Similarly, with the help of (2.13) and (2.14), from $(2.3)_2$, $(2.4)_2$, $(2.10)_2$ and the dual of $(2.5)_2$ we obtain

$$a^{\alpha\beta} = 2(I_1^2-I_2)^{-1}[I_1A^{\alpha\beta}-A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}]. \tag{2.15}$$

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We now recall the extrinsic properties of the surface. Let $b_{\alpha\beta}$ denote the coefficients of the second fundamental form of $\mathscr{s}$. Then,

$$b_{\alpha\beta} = -\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_{3,\beta} = \boldsymbol{a}_3 \cdot \boldsymbol{a}_{\alpha,\beta}, \quad b^\alpha_\beta = a^{\alpha\gamma} b_{\gamma\beta}, \tag{2.16}$$

$$2H = b^\alpha_\alpha = \frac{1}{r_1} + \frac{1}{r_2}, \quad K = \frac{1}{r_1 r_2}, \tag{2.17}$$

where H and K are the mean and the Gaussian curvatures of $\mathscr{s}$ and r_1 and r_2 denote the principal radii of curvature of the surface $\mathscr{s}$. For later reference, we also recall the formulae of Gauss and Mainardi–Codazzi. The latter given by

$$b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta} \tag{2.18}$$

involve only two independent equations corresponding to $\beta \neq \gamma$, while for $\beta = \gamma$ it is identically satisfied. Since the covariant surface curvature tensor has only one independent component, there is only one independent equation of Gauss for the surface $\mathscr{s}$ and this can be expressed as

$$R_{1212} = \det(b_{\alpha\beta}) = b_{11} b_{22} - b^2_{12} = aK, \tag{2.19}$$

or equivalently as

$$R_{\alpha\beta\gamma\delta} = K\epsilon_{\alpha\beta}\,\epsilon_{\gamma\delta}. \tag{2.20}$$

We denote the coefficient of the second fundamental form of the surface $\mathscr{S}$ by $B_{\alpha\beta}$, the principal radii of curvature of $\mathscr{S}$ by R_1, R_2 and the corresponding mean and Gaussian curvatures of the surface $\mathscr{S}$ by $\bar{H}$ and $\bar{K}$, respectively. Results similar to those of (2.16)–(2.20) hold also for the surface $\mathscr{S}$. In particular, we record the expressions

$$B_{\alpha\beta\dagger\gamma} = B_{\alpha\gamma\dagger\beta}, \quad \bar{R}_{1212} = \det(B_{\alpha\beta}) = A\bar{K}. \tag{2.21}$$

In the remainder of this section, we summarize the principal results from the nonlinear membrane theory of elastic shells for isotropic materials derived by direct approach. (For an account of the membrane theory by direct approach see § 14 of Naghdi (1972).) Let c be a closed curve on $\mathscr{s}$ and let

$$\boldsymbol{v} = \nu_\alpha \boldsymbol{a}^\alpha = \nu^\alpha \boldsymbol{a}_\alpha \tag{2.22}$$

be the outward unit normal to c lying in the surface. Further, let the tangential vector field $\boldsymbol{N}$, which depends on $\boldsymbol{v}$, represent the contact force (or the curve force vector) per unit length of c. Then, it can be shown that

$$\boldsymbol{N} = \boldsymbol{N}^\alpha \nu_\alpha = N^{\alpha\gamma} \nu_\alpha \boldsymbol{a}_\gamma, \tag{2.23}$$

where $N^{\alpha\gamma}$ are the components of $\boldsymbol{N}^\alpha$ referred to $\boldsymbol{a}_\gamma$. The local equation for conservation of mass can be expressed as

$$\rho a^{\frac{1}{2}} = \rho_0 A^{\frac{1}{2}} = k, \tag{2.24}$$

where ρ and ρ_0 denote the mass densities of the surface $\mathscr{s}$ and the reference surface $\mathscr{S}$, respectively, and k is a function of θ^α only. The equations of equilibrium are

$$\left.\begin{aligned} N^{\alpha\beta}{}_{|\alpha} + \rho f^\beta = 0, \quad b_{\alpha\beta} N^{\alpha\beta} + \rho f^3 = 0,\\ N^{\alpha\beta} = N^{\beta\alpha}, \end{aligned}\right\} \tag{2.25}$$

where $f^i = \boldsymbol{f} \cdot \boldsymbol{a}^i$ are the components of the assigned force $\boldsymbol{f}$ referred to the base vectors $\boldsymbol{a}_i$. It should be recalled that the assigned field $\boldsymbol{f}$ represents the combined effect of the stress vector on the major surfaces of the shell-like body, which we denote by $\boldsymbol{f}_c$, and a contribution arising from the three-dimensional body force $\boldsymbol{f}_b$ (see Naghdi 1972, § 11). In the present paper, we are concerned only with solutions of the membrane theory in the absence of body force so that $\boldsymbol{f}_b = 0$.

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Then, the components f^i in $(2.25)_{1,2}$ become $f^i = f^i_c$ and for later convenience we also introduce the notation

$$p = \rho f^3 = \rho f^3_c. \tag{2.26}$$

In view of the remarks preceding (2.26), it is clear that p and f^3 can be interpreted as the normal pressure per unit area of $\mathscr{s}$ and the normal pressure per unit mass of $\mathscr{s}$, respectively.

Within the scope of the purely mechanical theory, we may express the constitutive equations for $N^{\alpha\beta}$ in terms of a strain energy ψ per unit mass of $\mathscr{s}$. We may begin by assuming ψ to be a function of base vectors $\boldsymbol{a}_\alpha$ and their reference values in the form

$$\psi = \tilde{\psi}(\boldsymbol{a}_\alpha; \boldsymbol{A}_\alpha). \tag{2.27}$$

Then, by standard techniques we obtain‡

$$\psi = \overline{\psi}(a_{\alpha\beta}; A_{\alpha\beta}), \qquad N^{\alpha\beta} = \rho\left(\frac{\partial\overline{\psi}}{\partial a_{\alpha\beta}} + \frac{\partial\overline{\psi}}{\partial a_{\beta\alpha}}\right). \tag{2.28}$$

The above constitutive equations are valid for an elastic membrane which is anisotropic in the reference state. Assuming that the response function $\overline{\psi}$ is a polynomial in its arguments, then for an elastic membrane which initially is isotropic with a centre of symmetry, the strain energy density may be expressed as a different function of the joint invariants of $a_{\alpha\beta}$, $A_{\alpha\beta}$ as defined by (2.13). Hence

$$\psi = \hat{\psi}(I_1, I_2). \tag{2.29}$$

In order to express the constitutive equations for $N^{\alpha\beta}$ in terms of the function $\hat{\psi}$ for isotropic materials, we first record the following partial derivatives of the invariants (2.13), i.e.

$$\partial I_1/\partial a_{\alpha\beta} = A^{\alpha\beta}, \quad \partial I_2/\partial a_{\alpha\beta} = 2A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}. \tag{2.30}$$

Then, by chain rule differentiation, the constitutive equations $(2.28)_2$ in terms of $\hat{\psi}$ become

$$N^{\alpha\beta} = 2\rho[\Psi_1 A^{\alpha\beta} + 2\Psi_2 A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}], \tag{2.31}$$

where use has been made of (2.30) and we have set

$$\Psi_\alpha = \Psi_\alpha(I_1, I_2) = \partial\hat{\psi}/\partial I_\alpha \quad (\alpha = 1, 2). \tag{2.32}$$

3. Equivalence of results by direct approach and those derived from three-dimensional theory

The basic equations governing the equilibrium of a membrane by direct approach and the related constitutive results for a homogeneous and isotropic elastic membrane are summarized in § 2 (between equations (2.22) and (2.32)) in a form which is particularly convenient here. A derivation of corresponding results from the three-dimensional equations of nonlinear elasticity, including a detailed discussion of constitutive equations for both compressible and incompressible isotropic membranes, is given by Green & Adkins (1970). In the latter derivation, the response function for the strain energy density does not depend explicitly on the coefficients

‡ Instead of $(2.28)_1$, it is possible to assume that the function $\overline{\psi}$ depends also on the second fundamental form $B_{\alpha\beta}$ of the reference surface, but in the present paper the dependence of the strain energy on $B_{\alpha\beta}$ is excluded.

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$B_{\alpha\beta}$ of the second fundamental form in the reference configuration‡; and, as will become evident presently, 1–1 correspondence can be established between the field equations and the constitutive equations of this derivation and the corresponding results by direct approach given in § 2.

Consider a three-dimensional shell-like body $\mathscr{B}$ embedded in a Euclidean 3-space and identify the material points (or the particles) of the body with a system of convected coordinates θ^i; the convected coordinates are so chosen that they coincide with a normal coordinate system in the present deformed configuration with θ^α $(\alpha = 1, 2)$ on some reference surface, say the surface $\theta^3 = 0$, and with θ^3 along the normal to this surface. The boundary $\partial\mathscr{B}$ of the shell-like body in the deformed configuration consists of the major surfaces $\theta^3 = \pm h(\theta^\alpha)$ and the lateral surface $f(\theta^1, \theta^2) = 0$, where $2h(\theta^\alpha)$ denotes the thickness. Let $\boldsymbol{p}$ denote the position vector, relative to a fixed origin, of a typical particle of $\mathscr{B}$ in the present configuration. Then, $\boldsymbol{p}$ may be specified by

$$\boldsymbol{p} = \boldsymbol{r}(\theta^\alpha) + \theta^3 \boldsymbol{a}_3, \tag{3.1}$$

where $\boldsymbol{r}$ is the position vector of the middle surface $\theta^3 = 0$ designated as $\mathscr{s}$ and $\boldsymbol{a}_3$ is the unit normal to this surface.§ We denote the position vector, relative to the same fixed origin, of a typical particle of $\mathscr{B}$ in the initial undeformed configuration by $\boldsymbol{P}$ and write

$$\boldsymbol{P} = \boldsymbol{R}(\theta^\alpha) + (\theta^3/\lambda)\, \boldsymbol{A}_3 + O((\theta^3)^2), \tag{3.2}$$

where $\boldsymbol{R}$ is the position vector of the surface $\theta^3 = 0$ in the undeformed configuration designated as $\mathscr{S}$, $\boldsymbol{A}_3$ is the unit normal to $\mathscr{S}$, $\lambda = \lambda(\theta^\alpha)$ is the extension ratio (or the principal stretch) in the $\boldsymbol{A}_3$ direction and O stands for the usual order symbol, i.e. a function f_n is said to be $O(\epsilon_n)$ if there is a positive constant C such that $|f_n| \leqq C|\epsilon_n|$ for all sufficiently large n.

Let ρ^* and ρ_0^* be the mass densities in the deformed and the undeformed configurations of the shell-like body $\mathscr{B}$, respectively, and recall the local equation of conservation of mass in the form

$$\rho^* g^{\frac{1}{2}} = \rho_0^* G^{\frac{1}{2}} = k^*, \tag{3.3}$$

where

$$g = \det g_{ij}, \quad g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j, \quad \boldsymbol{g}_i = \partial \boldsymbol{p}/\partial\theta^i, \tag{3.4}$$

$$G = \det G_{ij}, \quad G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j, \quad \boldsymbol{G}_i = \partial \boldsymbol{P}/\theta^i \tag{3.5}$$

and k^* is a function of θ^i. The relation of the surface $\theta^3 = 0$ in the deformed configuration to the bounding surfaces $\theta^3 = \pm h(\theta^\alpha)$ is prescribed by

$$\int_{-h}^{h} k^* \theta^3 \,\mathrm{d}\theta^3 = 0. \tag{3.6}$$

Also, let ρ be the mass density per unit area of $\mathscr{s}$ and ρ_0 the mass density per unit area of $\mathscr{S}$ and define

$$\rho a^{\frac{1}{2}} = \int_{-h}^{h} \rho^* g^{\frac{1}{2}} \,\mathrm{d}\theta^3, \quad \rho_0 A^{\frac{1}{2}} = \int_{-h}^{h} \rho_0^* G^{\frac{1}{2}} \,\mathrm{d}\theta^3. \tag{3.7}$$

Then, the integration of (3.3) with respect to θ^3 between the limits $\theta^3 = -h, h$, yields the conservation of mass equation (2.24) provided we identify ρ and ρ_0 in (2.24) with $(3.7)_{1,2}$.

‡ This is in line with all existing developments of constitutive equations for various types of elastic shell theory from the three-dimensional equations (of which the membrane theory is one), where an assumption is introduced equivalent to the specification that the response function for the strain energy density be independent of the coefficients of the second fundamental form $B_{\alpha\beta}$ in the reference configuration.

§ Since later we identify the surface described by $\theta^3 = 0$ with the surface $\mathscr{s}$ of section 2, it is convenient to adopt the same notations such as (2.1) and (2.2) also in the present section.

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Let $a_{\alpha\beta}$, $a^{\alpha\beta}$ and $A_{\alpha\beta}$, $A^{\alpha\beta}$ stand for the components of the metric tensors of the surfaces s and $\mathscr{S}$, respectively, and let $2\bar{h}(\theta^\alpha)$ be the thickness in the undeformed reference configuration of $\mathscr{B}$. Further, let τ^{ij} denote the symmetric stress tensor per unit area in the deformed configuration of the body and recall that the stress resultants $N^{\alpha\beta}$ per unit length of the coordinates curves on s are defined by

$$\left.\begin{aligned} N^{\alpha\beta} &= \int_{-h}^{h} \mu\mu_\gamma^\beta \tau^{\gamma\alpha}\,\mathrm{d}\theta^3, \\ \mu_\gamma^\beta = \delta_\gamma^\beta - \theta^3 b_\gamma^\beta, &\quad \mu = \det \mu_\gamma^\beta = (g/a)^{\frac{1}{2}}, \end{aligned}\right\} \tag{3.8}$$

where b_γ^β is the second fundamental form of the surface s.

In what follows, we confine attention to the case in which the external surface loads are along the normals to the major surfaces $\theta^3 = \pm h(\theta^\alpha)$. Then, the equilibrium equations for the membrane are of the same forms as those given by (2.25) with $f^\beta = 0$ provided $N^{\alpha\beta}$ is identified with $(3.8)_1$ and

$$p\boldsymbol{n}a^{\frac{1}{2}} = [\boldsymbol{T}^3 - h_{,\alpha}\,\boldsymbol{T}^\alpha]_{\theta^3=h} - [\boldsymbol{T}^3 - h_{,\alpha}\,\boldsymbol{T}^\alpha]_{\theta^3=-h}, \tag{3.9}$$

where

$$\boldsymbol{T}^i = g^{\frac{1}{2}}\tau^{ij}\boldsymbol{g}_j \tag{3.10}$$

and

$$\boldsymbol{n} = \frac{\boldsymbol{p}_{,1}\times\boldsymbol{p}_{,2}}{|\boldsymbol{p}_{,1}\times\boldsymbol{p}_{,2}|}, \quad \boldsymbol{p}_{,\alpha} = \mu_\alpha^\gamma\,\boldsymbol{a}_\gamma + h_{,\alpha}\,\boldsymbol{a}_3. \tag{3.11}$$

The basic equations of the membrane theory derived from the three-dimensional equations consist of the kinematic expressions (3.1), (3.2) and the equilibrium equations (2.25) with $f^\beta = 0$ and with $\rho f^3 = p$ given by (3.9), in addition to appropriate constitutive equations. For an initially homogeneous medium for which the mass density $\rho_0^* = \text{const.}$, the constitutive equations for an isotropic elastic membrane (essentially in the notation of Green & Adkins 1970) are

$$\left.\begin{aligned} N^{\alpha\beta} &= 2\bar{h}\lambda\{(\Phi+\lambda^2\Psi)\,A^{\alpha\beta} + [(J_3\lambda^{-2}+\lambda^4-\lambda^2 J_1)\,\Psi - \lambda^2\Phi]\,a^{\alpha\beta}\}, \\ &\lambda^2\Phi + \lambda^2(J_1-\lambda^2)\,\Psi + P = 0, \\ &h = \lambda\bar{h}. \end{aligned}\right\} \tag{3.12}$$

For an elastic material which is compressible (in the context of the three-dimensional theory), the functions Φ, Ψ, P are related to a strain energy density W, per unit volume in the initial undeformed configuration, by

$$\left.\begin{aligned} W &= \hat{W}(J_1, J_2, J_3), \\ \Phi = 2(J_3)^{-\frac{1}{2}}\,\partial\hat{W}/\partial J_1, &\quad \Psi = 2(J_3)^{-\frac{1}{2}}\,\partial\hat{W}/\partial J_2, \\ P &= 2(J_3)^{\frac{1}{2}}\,\partial\hat{W}/\partial J_3, \end{aligned}\right\} \tag{3.13}$$

where the strain invariants J_m $(m = 1, 2, 3)$ are

$$J_1 = \lambda^2 + A^{\alpha\beta}a_{\alpha\beta}, \quad J_2 = (a/A)\,(1+\lambda^2 A_{\alpha\beta}\,a^{\alpha\beta}), \quad J_3 = \lambda^2 a/A \tag{3.14}$$

and $A = \det A_{\alpha\beta}$. In the compressible case, the strain energy W is a function of all three invariants (3.14). Moreover, given the results (3.13), the equation $(3.12)_2$ may be regarded as the equation for the determination of the extension ratio λ. On the other hand, for an incompressible material, the incompressibility condition $J_3 = 1$ implies that $\lambda^2 a/A = 1$ and the scalar function P is no longer given by a constitutive equation. In the latter incompressible case, corresponding to (3.13), we have

$$\left.\begin{aligned} W &= \tilde{W}(J_1, J_2), \\ \Phi = 2\partial\tilde{W}/\partial J_1, &\quad \Psi = 2\partial\tilde{W}/\partial J_2 \end{aligned}\right\} \tag{3.15}$$

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and the strain invariants are now reduced to

$$J_1 = \lambda^2 + A^{\alpha\beta} a_{\alpha\beta}, \quad J_2 = (1/\lambda^2)\, A_{\alpha\beta}\, a^{\alpha\beta}, \quad J_3 = \lambda^2 a/A = 1. \tag{3.16}$$

As is evident from $(3.15)_1$, in the case of an incompressible material, the strain energy depends only on J_1, J_2 and the extension ratio is obtained from the contraint condition $(3.16)_3$ and the equation $(3.12)_2$ can then be used to determine the scalar function P.

Now in order to compare the above constitutive equations with the corresponding expressions by direct approach (see equations (2.29), (2.31) and (2.32)), we need to express $(3.12)_1$ in terms of the same variables as those in (2.31) and, in particular, eliminate λ^2 from $(3.12)_1$. In the incompressible case, by use of $(3.16)_3$, $(2.13)_1$, $(3.15)_{2,3}$ and the identities (2.14) and (2.15), the expressions $(3.12)_1$ and $(3.16)_{1,2}$ can be reduced to

$$N^{\alpha\beta} = 4\bar{h}\left(\frac{A}{a}\right)^{\frac{1}{2}} \left\{[1 - 4I_1(I_1^2 - I_2)^{-2}]\frac{\partial \tilde{W}}{\partial J_1} + [2(I_1^2 - I_2)^{-1} - 4I_1^2(I_1^2 - I_2)^{-2} + I_1]\frac{\partial \tilde{W}}{\partial J_2}\right\} A^{\alpha\beta}$$
$$+ 8\bar{h}\left(\frac{A}{a}\right)^{\frac{1}{2}} \left\{[2(I_1^2 - I_2)^{-2}]\frac{\partial \tilde{W}}{\partial J_1} + [2I_1(I_1^2 - I_2)^{-2} - \tfrac{1}{2}]\frac{\partial \tilde{W}}{\partial J_2}\right\} A^{\alpha\xi} A^{\beta\eta} a_{\xi\eta} \tag{3.17}$$

and
$$J_1 = 2(I_1^2 - I_2)^{-1} + I_1, \quad J_2 = 2I_1(I_1^2 - I_2)^{-1} + \tfrac{1}{2}(I_1^2 - I_2), \tag{3.18}$$

respectively. In view of (3.18), the response function $\tilde{W}$ in $(3.11)_1$ may be regarded as a different function $\overline{W}$ of I_1, I_2 and hence we may write

$$W = \tilde{W}(J_1, J_2) = \overline{W}(I_1, I_2). \tag{3.19}$$

Further, using the chain rule for differentiation, we obtain from (3.19) and (3.18) that

$$\left.\begin{aligned} \frac{\partial \overline{W}}{\partial I_1} &= [1 - 4I_1(I_1^2 - I_2)^{-2}]\frac{\partial \tilde{W}}{\partial J_1} + [2(I_1^2 - I_2)^{-1} - 4I_1^2(I_1^2 - I_2)^{-2} + I_1]\frac{\partial \tilde{W}}{\partial J_2}, \\ \frac{\partial \overline{W}}{\partial I_2} &= [2(I_1^2 - I_2)^{-2}]\frac{\partial \tilde{W}}{\partial J_1} + [2I_1(I_1^2 - I_2)^{-2} - \tfrac{1}{2}]\frac{\partial \tilde{W}}{\partial J_2}, \end{aligned}\right\} \tag{3.20}$$

so that
$$N^{\alpha\beta} = 4\bar{h}\left(\frac{A}{a}\right)^{\frac{1}{2}} \left\{\frac{\partial \overline{W}}{\partial I_1} A^{\alpha\beta} + 2\frac{\partial \overline{W}}{\partial I_2} A^{\alpha\xi} A^{\beta\eta} a_{\xi\eta}\right\}. \tag{3.21}$$

In order to put (3.21) in a form which would permit ready comparison with a corresponding constitutive equation obtained by direct approach, we need an explicit expression relating the density ρ_0 to the reference mass density ρ_0^* of $\mathscr{B}$. To this end, following a customary procedure for membranes and *thin* shells, we omit terms of $O((\theta^3)^2)$ in the expansion (3.2) and adopt the approximation for $\boldsymbol{P}$ which consists of only the first two terms on the right-hand side of (3.2). In view of the usual assumption employed in the development of *thin* shell theory, namely the assumption that $(\bar{h}/R) \ll 1$, R being the minimum radius of the curvature of the surface $\theta^3 = 0$ in the undeformed configuration, it follows from (3.6) that the mass density $\rho_0^* \simeq k^*/A^{\frac{1}{2}}$ is independent of θ^3 and the condition (3.6) is satisfied to the order of approximation considered (for details see Naghdi 1972, p. 472). We then find that $(3.7)_2$ can be approximated by $\rho_0 = 2\bar{h}\rho_0^*$ so that‡

$$\rho = 2\bar{h}(A/a)^{\frac{1}{2}} \rho_0^*. \tag{3.22}$$

‡ The expression (3.22) is, of course, applicable to both incompressible and compressible materials.

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Using the last result and introducing the strain energy density Σ per unit mass ρ_0^* of the body $\mathscr{B}$ in terms of the response function $\tilde{\Sigma}$, namely

$$\Sigma = \frac{W}{\rho_0^*} = \tilde{\Sigma}(I_1, I_2), \tag{3.23}$$

we find that the constitutive equation (3.21) becomes

$$N^{\alpha\beta} = 2\rho\left[\frac{\partial\tilde{\Sigma}}{\partial I_1}A^{\alpha\beta} + 2\frac{\partial\tilde{\Sigma}}{\partial I_2}A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}\right], \tag{3.24}$$

which has the same form as (2.31).

When the material is compressible, λ^2 cannot be eliminated immediately as in the incompressible case. However, with the help of $(2.13)_1$, (2.14), (2.15), (3.13) and $(3.14)_3$, the constitutive equations $(3.12)_{1,2}$ and the strain invariants J_m in (3.14) can be expressed in terms of I_1, I_2 and λ^2 as follows:

$$\left.\begin{aligned} N^{\alpha\beta} &= 4\bar{h}\left(\frac{A}{a}\right)^{\frac{1}{2}}\left[\frac{\partial\hat{W}}{\partial J_1} + (\lambda^2 + I_1)\frac{\partial\hat{W}}{\partial J_2} - 2\lambda^2 I_1 (I_1^2 - I_2)^{-1}\left(\frac{\partial\hat{W}}{\partial J_1} + I_1\frac{\partial\hat{W}}{\partial J_2}\right)\right]A^{\alpha\beta} \\ &\quad + 8\bar{h}\left(\frac{A}{a}\right)^{\frac{1}{2}}\left[-\frac{1}{2}\frac{\partial\hat{W}}{\partial J_2} + \lambda^2(I_1^2 - I_2)^{-1}\left(\frac{\partial\hat{W}}{\partial J_1} + I_1\frac{\partial\hat{W}}{\partial J_2}\right)\right]A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}, \\ &2(I_1^2 - I_2)^{-1}\left(\frac{\partial\hat{W}}{\partial J_1} + I_1\frac{\partial\hat{W}}{\partial J_2}\right) + \frac{\partial\hat{W}}{\partial J_3} = 0 \end{aligned}\right\} \tag{3.25}$$

and

$$J_1 = I_1 + \lambda^2, \quad J_2 = \tfrac{1}{2}(I_1^2 - I_2) + \lambda^2 I_1, \quad J_3 = \tfrac{1}{2}\lambda^2(I_1^2 - I_2). \tag{3.26}$$

With the use of $(3.25)_2$, the constitutive equation $(3.25)_1$ can be expressed in the following compact form:

$$N^{\alpha\beta} = 4\bar{h}\left(\frac{A}{a}\right)^{\frac{1}{2}}\left[\frac{\partial\hat{W}}{\partial J_1} + (\lambda^2 + I_1)\frac{\partial\hat{W}}{\partial J_2} + \lambda^2 I_1\frac{\partial\hat{W}}{\partial J_3}\right]A^{\alpha\beta} + 8\bar{h}\left(\frac{A}{a}\right)^{\frac{1}{2}}\left[-\frac{1}{2}\frac{\partial\hat{W}}{\partial J_2} - \tfrac{1}{2}\lambda^2\frac{\partial\hat{W}}{\partial J_3}\right]A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}. \tag{3.27}$$

In view of (3.26), the response function $\hat{W}$ in (3.13) may be regarded as a different function W' of I_1, I_2 and λ^2. Hence, we may write

$$W = \hat{W}(J_1, J_2, J_3) = W'(I_1, I_2, \lambda^2). \tag{3.28}$$

Further, using the chain rule for differentiation, from (3.28) and (3.26) we obtain

$$\left.\begin{aligned} \frac{\partial W'}{\partial I_1} &= \frac{\partial\hat{W}}{\partial J_1} + (I_1 + \lambda^2)\frac{\partial\hat{W}}{\partial J_2} + I_1\lambda^2\frac{\partial\hat{W}}{\partial J_3}, \\ \frac{\partial W'}{\partial I_2} &= -\frac{1}{2}\frac{\partial\hat{W}}{\partial J_2} - \frac{\lambda^2}{2}\frac{\partial\hat{W}}{\partial J_3}, \\ \frac{\partial W'}{\partial\lambda^2} &= \frac{\partial\hat{W}}{\partial J_1} + I_1\frac{\partial\hat{W}}{\partial J_2} + \tfrac{1}{2}(I_1^2 - I_2)\frac{\partial\hat{W}}{\partial J_3}. \end{aligned}\right\} \tag{3.29}$$

Comparison of $(3.29)_3$ with $(3.25)_2$ at once yields

$$\partial W'/\partial\lambda^2 = 0, \tag{3.30}$$

so that in terms of the strain energy density response function $\hat{\Sigma}$ per unit mass ρ_0^* of $\mathscr{B}$ we have

$$\Sigma = W'/\rho_0^* = \hat{\Sigma}(I_1, I_2) \tag{3.31}$$

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and the constitutive equation (3.27) can be rewritten as

$$N^{\alpha\beta} = 2\rho\left[\frac{\partial\hat{\Sigma}}{\partial I_1}A^{\alpha\beta} + 2\frac{\partial\hat{\Sigma}}{\partial I_2}A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}\right], \tag{3.32}$$

which is of the same form as (2.31).

The membrane theory discussed above, after elimination of λ^2, is characterized by a system of two-dimensional equations which consist of the field equations (2.24) and (2.25) with $f^\beta = 0$, as well as the constitutive equations (3.23) and (3.24) for the incompressible material or (3.31) and (3.32) for the compressible case. If we identify $\boldsymbol{r}$ and $\boldsymbol{a}_3$ in (3.1), as well as $\boldsymbol{R}$ and $\boldsymbol{A}_3$ in the approximation for the position vector $\boldsymbol{P}$ introduced above [following equation (3.21)], with corresponding quantities of the direct development summarized in § 2, then the two developments are formally equivalent. In particular, comparison of the field equations and the constitutive equations reveals a 1–1 correspondence between the two systems of equations provided we identify: (i) the densities ρ and ρ_0 in (2.24) with the expressions $(3.7)_{1,2}$; (ii) the normal component of the assigned force per unit area of $\mathscr{s}$ given by (2.26) with the net normal surface pressure p in (3.9); the components $N^{\alpha\beta}$ in (2.25) with the stress resultants $(3.8)_1$; and the strain energy response function $\hat{\psi}$ in (2.29) either with the response function $\tilde{\Sigma}$ for the incompressible material or with the function $\hat{\Sigma}$ in the compressible case.

4. Formulation of the problem

We are concerned here with large deformation solutions possible in every isotropic elastic membrane whose strain energy response function is specified by (2.29). For brevity, in what follows, we often refer to such solutions as controllable. Recalling (from § 2) that the surface $\mathscr{S}$ in the initial undeformed configuration becomes the surface $\mathscr{s}$ in the deformed configuration of the membrane, we assume that the initial mass density ρ_0 is uniform throughout $\mathscr{S}$, i.e.‡

$$\rho_0 = \text{const.} \tag{4.1}$$

and that the external surface load acting on $\mathscr{s}$ is a uniform normal pressure alone. Thus, in the absence of body force, the assigned fields in $(2.25)_{1,2}$ and (2.26) are

$$f^\beta = 0, \quad f^3 = \text{const.} \tag{4.2}$$

Let the coefficients of the first fundamental forms $a_{\alpha\beta}(\theta^\gamma)$, $A_{\alpha\beta}(\theta^\gamma)$ be positive definite, symmetric, real-valued tensor functions of class C^2 and let the coefficients of the second fundamental forms $b_{\alpha\beta}(\theta^\gamma)$, $B_{\alpha\beta}(\theta^\gamma)$ be symmetric real-valued functions of class C^1. If the set of variables

$$A_{\alpha\beta}(\theta^\gamma), \quad B_{\alpha\beta}(\theta^\gamma) \tag{4.3}$$

of the reference surface $\mathscr{S}$ and

$$a_{\alpha\beta}(\theta^\gamma), \quad b_{\alpha\beta}(\theta^\gamma) \tag{4.4}$$

of the deformed surface $\mathscr{s}$ describe the desired controllable deformation resulting from a uniform normal pressure (4.2), as well as appropriate edge loads, then (4.3) and (4.4) must satisfy the continuity equation (2.24), the equilibrium equations (2.25) with ρ_0, f^β, f^3 specified by (4.1) and (4.2) and the compatibility equations $(2.21)_{1,2}$, (2.18) and (2.19) for every choice of the response function $\hat{\psi}$ in (2.29). These equations should yield the least restrictions to be imposed

‡ The assumption (4.1) is a mild restriction in the present development. However, our reason for introducing this assumption will become clear later in § 5 [see the discussion preceding (5.10)].

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on the functions (4.3) and (4.4) in order to obtain all controllable solutions of the type under consideration.

The restrictions demanded by the equilibrium equations are obtained by substituting the constitutive equations (2.31) for $N^{\alpha\beta}$ into the differential equations (2.25). Thus,

$$\left.\begin{aligned}\sum_{r=1}^{2}\Psi_r\left[2\rho\frac{\partial I_r}{\partial a_{\alpha\beta}}\right]_{|\alpha}+\sum_{r,s=1}^{2}\Psi_{rs}\left[2\rho\frac{\partial I_r}{\partial a_{\alpha\beta}}I_{s,\alpha}\right]&=-\rho f^{\beta},\\ \sum_{r=1}^{2}\Psi_r\left[2b_{\alpha\beta}\frac{\partial I_r}{\partial a_{\alpha\beta}}\right]&=-f^{3},\end{aligned}\right\}\tag{4.5}$$

where $\partial I_r/\partial a_{\alpha\beta}$ ($r = 1, 2$), are given in (2.30), the functions Ψ_γ are defined by (2.32) and

$$\Psi_{rs}=\frac{\partial^2\hat{\psi}(I_1,I_2)}{\partial I_r\,\partial I_s}\quad(r,s=1,2).\tag{4.6}$$

Observing that in $(4.5)_2$ the assigned field f^3 = constant if and only if $f^3_{,\sigma} = 0$, with the use of (4.2) and the continuity equation (2.24), from (4.5) we deduce

$$\left.\begin{aligned}\sum_{r=1}^{2}\Psi_r\left[\left(\frac{A}{a}\right)^{\frac{1}{2}}\frac{\partial I_r}{\partial a_{\alpha\beta}}\right]_{|\alpha}+\sum_{r,s=1}^{2}\Psi_{rs}\left[\left(\frac{A}{a}\right)^{\frac{1}{2}}\frac{\partial I_r}{\partial a_{\alpha\beta}}I_{s,\alpha}\right]&=0,\\ \sum_{r=1}^{2}\Psi_r\left[b_{\alpha\beta}\frac{\partial I_r}{\partial a_{\alpha\beta}}\right]_{,\sigma}+\sum_{r,s=1}^{2}\Psi_{rs}\left[b_{\alpha\beta}\frac{\partial I_r}{\partial a_{\alpha\beta}}I_{s,\sigma}\right]&=0,\end{aligned}\right\}\tag{4.7}$$

where we have also made use of the assumption (4.1). The above two conditions involve the kinematic variables (4.3) and (4.4) and the material properties through the response function $\hat{\psi}$ and its partial derivatives.

In order that (4.7) be satisfied by the variables (4.3) and (4.4) for every choice of $\hat{\psi}$, it is necessary and sufficient that the coefficient of each distinct derivatives of $\hat{\psi}$ in (4.7) vanish independently. Equating to zero these coefficients and simplifying the resulting expressions with the help of $(2.9)_1$, $(2.13)_{1,2}$, (2.30) and using the fact that, with $A^{\alpha\beta}$ nonsingular, $A^{\alpha\beta}I_{s,\beta} = 0$ ($s = 1, 2$) is equivalent to $I_{s,\beta} = 0$ or simply I_s = constant, we deduce the following six restrictions:

$$\left.\begin{aligned}I_1=A^{\alpha\beta}a_{\alpha\beta}=c_1,&\quad I_2=A^{\alpha\xi}A^{\beta\eta}a_{\alpha\beta}a_{\xi\eta}=c_2,\\ \left[\left(\frac{A}{a}\right)^{\frac{1}{2}}A^{\alpha\beta}\right]_{|\alpha}=0,&\quad\left[\left(\frac{A}{a}\right)^{\frac{1}{2}}A^{\alpha\xi}A^{\beta\eta}a_{\xi\eta}\right]_{|\alpha}=0,\\ b_{\alpha\beta}A^{\alpha\beta}=c_3,&\quad b_{\alpha\beta}a_{\xi\eta}A^{\alpha\xi}A^{\beta\eta}=c_4,\end{aligned}\right\}\tag{4.8}$$

where $c_1, \ldots, c_4$ are constants. Since these constants must be regarded as prescribed for controllable deformations under discussion, henceforth they will be referred to as the prescribed deformation parameters.‡

By expanding the left-hand sides of $(4.8)_{1,2}$ and making use of the metric properties of both $a_{\alpha\beta}$ and $A_{\alpha\beta}$, including $(2.10)_3$ along with

$$A_{\alpha\alpha}>0,\quad a_{\beta\beta}>0\ (\text{no sum on }\alpha,\beta)\tag{4.9}$$

and forming the identity $2I_2 - I_1^2$ or equivalently

$$2c_2-c_1^2=\frac{A_{11}A_{22}}{A^2}\left\{A\left(\frac{a_{11}}{A_{11}}-\frac{a_{22}}{A_{22}}\right)^2+\left[2a_{12}-A_{12}\left(\frac{a_{11}}{A_{11}}+\frac{a_{22}}{A_{22}}\right)^2\right]\right\},\tag{4.10}$$

it can be shown that

$$c_1>0,\quad 2c_2-c_1^2\geqq 0,\tag{4.11}$$

‡ It will become evident later that not all of these constants are independent.

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where in obtaining the last inequality we have also used the fact that $a_{\alpha\beta}$, $A_{\alpha\beta}$ are reals. Moreover, with the help of (2.14), $(2.1)_3$ and $(2.10)_3$, subtraction of $(4.8)_2$ from the square of $(4.8)_1$ results in the inequality

$$a/A = \tfrac{1}{2}(c_1^2 - c_2) > 0. \tag{4.12}$$

Also, from (4.12) and $(4.11)_2$ we have the result

$$c_1^2 > 2c_2 - c_1^2 \geqslant 0. \tag{4.13}$$

The deformation parameters c_3 and c_4 must be further restricted when the controllable deformation is sustained by edge loads alone. These restrictions are specified in the following

Lemma 4.1. *The specification* $\boldsymbol{f} = \mathbf{0}$, *i.e. the absence of surface load, in a controllable deformation implies the vanishing of the deformation parameters* c_3 *and* c_4 *and conversely*:

$$\boldsymbol{f} = \mathbf{0} \Leftrightarrow c_3 = c_4 = 0. \tag{4.14}$$

Proof. In view of (4.2), we only need to show that $f^3 = 0$ if and only if $c_3 = c_4 = 0$. If $f^3 = 0$, the right-hand side of $(4.5)_2$ is zero; and, on the left-hand side of this equation, the coefficients Ψ_r $(r = 1, 2)$ must vanish independently for controllable deformations. But, by $(2.30)_{1,2}$ and $(4.8)_{1,2}$, these coefficients are c_3 and c_4 and this proves the 'if' part of the lemma. The 'only if' part can be easily established by using the converse argument.

We close this section by making an additional observation concerning the nature of the specification of the surface load. The equilibrium conditions in (4.8) have been deduced when the surface load is a uniform normal pressure per unit mass as specified by (4.2). The same equilibrium conditions (4.8) can be obtained if the surface load is specified by a uniform normal pressure per unit area, i.e. by

$$f^\beta = 0, \quad p = \text{const.} \tag{4.15}$$

Hence, all of the results derived in this paper are valid regardless of whether the uniform normal pressure is measured per unit mass or per unit area of the deformed surface $\mathscr{s}$.

5. Solutions for the first and second fundamental forms of the deformed membrane

The equilibrium conditions in (4.8), along with the compatibility equations $(2.21)_{1,2}$, (2.18) and (2.19), characterize the restrictions for obtaining the desired controllable solutions when f^β and f^3 are specified by (4.2). The first four conditions of (4.8) consisting of two algebraic equations $(4.8)_{1,2}$ and four scalar partial differential equations $(4.8)_{3,4}$ are restrictions on the metric coefficients $a_{\alpha\beta}$ of the deformed surface $\mathscr{s}$. Inasmuch as these equations do not involve the coefficients of the second fundamental form $b_{\alpha\beta}$ of the deformed surface, we first solve the system of equations $(4.8)_{1,2,3,4}$ for $a_{\alpha\beta}$ in terms of the initial metric tensor $A_{\alpha\beta}$ and the prescribed deformation parameters c_1, c_2.

Solution for $a_{\alpha\beta}$. Recalling that the left-hand side of (4.12) was obtained from combination of $(4.8)_{1,2}$ and that c_1, c_2 are constants, we differentiate (4.12) with respect to θ^α and utilize $(2.8)_3$ and its dual to obtain

$${}_d\Gamma^\sigma_{\sigma\alpha} = 0, \tag{5.1}$$

where the temporary notation ${}_d\Gamma^\gamma_{\alpha\beta}$ for the 'difference' Christoffel symbol is defined by

$${}_d\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \bar{\Gamma}^\gamma_{\alpha\beta}. \tag{5.2}$$

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Making use of (4.12), $(2.9)_1$, $(2.6)_2$, the dual of $(2.8)_2$ and (5.1), the restrictions $(4.8)_3$ and $(4.8)_4$ can be reduced to

$$A^{\alpha\gamma}\,{}_d\Gamma^{\beta}_{\alpha\gamma} = 0 \tag{5.3}$$

and

$$A^{\alpha\xi}a_{\xi\eta}[{}_d\Gamma^{\eta}_{\alpha\sigma}A^{\sigma\beta} + {}_d\Gamma^{\beta}_{\alpha\sigma}A^{\sigma\eta}] = 0, \tag{5.4}$$

respectively.

It should be apparent that (5.1), (5.3) and (5.4) represent a system of six simultaneous homogeneous algebraic equations in the six unknown variables ${}_d\Gamma^{\gamma}_{\alpha\beta}$. This system of equations has possible nontrivial solution only if the determinant of the coefficients of ${}_d\Gamma^{\gamma}_{\alpha\beta}$ in (5.1), (5.3) and (5.4), namely

$$\Delta = -\frac{4A_{11}A_{22}}{A^4}\left\{A\left(\frac{a_{11}}{A_{11}}-\frac{a_{22}}{A_{22}}\right)^2+\left[2a_{12}-A_{12}\left(\frac{a_{11}}{A_{11}}+\frac{a_{22}}{A_{22}}\right)\right]^2\right\} \tag{5.5}$$

vanishes. Since each of the two squared quantities in (5.5) is real and independent and since $A > 0$, for the singular case in which $\Delta = 0$ we have

$$\frac{a_{11}}{A_{11}} = \frac{a_{22}}{A_{22}}, \quad a_{12} = \tfrac{1}{2}A_{12}\left(\frac{a_{11}}{A_{11}}+\frac{a_{22}}{A_{22}}\right). \tag{5.6}$$

With the help of $(4.8)_1$, the dual of $(2.4)_2$ and (5.6) we then obtain

$$a_{\alpha\beta} = \tfrac{1}{2}c_1 A_{\alpha\beta} \quad (c_1 > 0). \tag{5.7}$$

In view of the simple relation (5.7), it follows from the definition (2.7) and $(2.4)_2$ and their duals, as well as (5.2), that

$${}_d\Gamma^{\gamma}_{\alpha\beta} = 0 \quad \text{or} \quad \Gamma^{\gamma}_{\alpha\beta} = \bar{\Gamma}^{\gamma}_{\alpha\beta} \tag{5.8}$$

trivially satisfies the system of equations (5.1), (5.3) and (5.4). It may be emphasized that the above solution for the singular case is obtained with the help of $(4.8)_1$ and hence (5.8) should be regarded as a solution of $(4.8)_1$, (5.1), (5.3) and (5.4).

We consider now the system of equations (5.1), (5.3) and (5.4) when Δ does not vanish. Since homogeneous algebraic equations with nonzero determinant of the coefficients have trivial solutions, it follows at once that (5.8) is also a solution in this case. To show that the solution (5.8) also satisfies $(4.8)_1$ in the nonsingular case ($\Delta \neq 0$), we differentiate $(4.8)_1$ covariantly (with respect to $a_{\alpha\beta}$) and then by a procedure similar to that which led to (5.4), write the resulting expression in the form

$$a_{\alpha\beta}[{}_d\Gamma^{\alpha}_{\sigma\gamma}A^{\beta\gamma} + {}_d\Gamma^{\beta}_{\sigma\gamma}A^{\alpha\gamma}] = 0, \tag{5.9}$$

which is identically satisfied by (5.8). Hence, (5.8) is also a solution of $(4.8)_1$, (5.1), (5.3) and (5.4) in the nonsingular case.

With reference to the system of equations which led to the solution (5.8), it is worth observing here that in the singular case for which the determinant Δ in (5.5) vanishes, not all the six equations (5.1), (5.3) and (5.4) are linearly independent and we need to have $(4.8)_1$ in order to obtain the result (5.8). In the nonsingular case ($\Delta \neq 0$) on the other hand, we have the seven equations (5.1), (5.3), (5.4) and (4.8); but, as indicated above, the latter equation $(4.8)_1$ is redundant when $\Delta \neq 0$.

The above observation also bears on the assumption (4.1). Without this assumption, the conditions corresponding to $(4.8)_{1,2,3,4}$ will also involve terms of the type $\rho_{0,\alpha}/\rho_0$ and the counterparts of (5.1), (5.3), (5.4) and $(4.8)_1$ will lead to a system of equations in the variables $\rho_{0,\alpha}$ and ${}_d\Gamma^{\gamma}_{\alpha\beta}$. The determinant of the coefficients of this system of equations is very complex to render a solution and this is the main reason for our assumption (4.1).

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We now return to (5.8) and proceed to complete our solution of $a_{\alpha\beta}$. Thus, with the use of $(2.8)_1$ and (2.7), we first replace $(5.8)_2$ by the equivalent relation‡

$$a_{\alpha\beta,\gamma} = a_{\alpha\sigma}\bar{\Gamma}^{\sigma}_{\beta\gamma} + a_{\beta\gamma}\bar{\Gamma}^{\sigma}_{\alpha\gamma}, \tag{5.10}$$

which represents a system of nonlinear partial differential equations of the first order in $a_{\alpha\beta}$. Since $a_{\alpha\beta}$ is assumed to be of class C^2, we have the integrability conditions

$$a_{\alpha\beta,\gamma\delta} = a_{\alpha\beta,\delta\gamma}, \tag{5.11}$$

which are both necessary and sufficient for the existence of solutions of (5.10). Next, substitute (5.10) into (5.11) to obtain

$$a_{\alpha\nu}(\bar{\Gamma}^{\nu}_{\beta\gamma,\delta} - \bar{\Gamma}^{\nu}_{\beta\delta,\gamma}) + a_{\beta\nu}(\bar{\Gamma}^{\nu}_{\alpha\gamma,\delta} - \bar{\Gamma}^{\nu}_{\alpha\delta,\gamma}) + a_{\alpha\nu}(\bar{\Gamma}^{\nu}_{\sigma\delta}\bar{\Gamma}^{\sigma}_{\beta\gamma} - \bar{\Gamma}^{\nu}_{\sigma\gamma}\bar{\Gamma}^{\sigma}_{\beta\delta}) + a_{\beta\nu}(\bar{\Gamma}^{\nu}_{\sigma\delta}\bar{\Gamma}^{\sigma}_{\alpha\gamma} - \bar{\Gamma}^{\nu}_{\sigma\gamma}\bar{\Gamma}^{\sigma}_{\alpha\delta}) = 0. \tag{5.12}$$

With the help of $(2.10)_{4,6}$ and the dual of $(2.8)_2$, the terms in (5.12) which involve the partial derivatives of the Christoffel symbols of the second kind such as $\bar{\Gamma}^{\sigma}_{\beta\gamma}$ can be expressed in terms of the partial derivatives of the Christoffel symbols of the first kind, i.e.,

$$\bar{\Gamma}^{\sigma}_{\beta\gamma,\delta} = A^{\sigma\nu}(\bar{\Gamma}_{\beta\gamma\nu,\delta} - \bar{\Gamma}^{\lambda}_{\beta\gamma}\bar{\Gamma}_{\nu\delta\lambda}) - \bar{\Gamma}^{\epsilon}_{\beta\gamma}\bar{\Gamma}^{\sigma}_{\epsilon\delta}. \tag{5.13}$$

Using the last result and recalling the formula (2.11) for the curvature tensor $\bar{R}_{\lambda\alpha\beta\gamma}$, we can reduce (5.12) to

$$a_{\alpha\mu}A^{\mu\nu}\bar{R}_{\nu\beta\delta\gamma} + a_{\beta\mu}A^{\mu\nu}\bar{R}_{\nu\alpha\delta\gamma} = 0, \tag{5.14}$$

or equivalently in terms of the Gaussian curvature of $\mathscr{S}$ to

$$\bar{K}(a_{\alpha\mu}A^{\mu\nu}\bar{\epsilon}_{\nu\beta} + a_{\beta\mu}A^{\mu\nu}\bar{\epsilon}_{\nu\alpha}) = 0, \tag{5.15}$$

where in obtaining (5.15) use has been made also of the dual of (2.20) and $(2.10)_1$. Since the last two results are obtained as a consequence of the integrability condition (5.11), it follows that the functions $a_{\alpha\beta}$ and $A_{\alpha\beta}$ on the right-hand side of (5.10) must be restricted by either (5.14) or (5.15). If the Gaussian curvature $\bar{K}$ is zero, the condition (5.15) is automatically fulfilled. On the other hand, if $\bar{K}$ does not vanish, the functions $a_{\alpha\beta}$ and $A_{\gamma\delta}$ in addition to $(5.8)_2$ must also satisfy the three relations

$$a_{\alpha\mu}A^{\mu\nu}\bar{\epsilon}_{\nu\beta} + a_{\beta\mu}A^{\mu\nu}\bar{\epsilon}_{\nu\alpha} = 0. \tag{5.16}$$

It is therefore clear that we need to consider two cases according to whether or not the Gaussian curvature $\bar{K}$ is zero:

Case (*I*): $\bar{K} = 0$. In this case, since (5.15) is satisfied, the system of differential equations (5.10) is completely integrable in the sense of Eisenhart (1941, §23) and a solution for $a_{\alpha\beta}$ can be obtained in the form §

$$\left.\begin{aligned} \frac{a_{11}}{A_{11}} &= \tfrac{1}{2}c_1\left\{1 \pm \left(\frac{2c_2}{c_1^2} - 1\right)^{\frac{1}{2}}(A_{11}A_{22})^{-\frac{1}{2}}[A_{12}\cos V + A^{\frac{1}{2}}\sin V]\right\},\\ \frac{a_{22}}{A_{22}} &= \tfrac{1}{2}c_1\left\{1 \pm \left(\frac{2c_2}{c_1^2} - 1\right)^{\frac{1}{2}}(A_{11}A_{22})^{-\frac{1}{2}}[A_{12}\cos V - A^{\frac{1}{2}}\sin V]\right\},\\ a_{12} &= \tfrac{1}{2}c_1\left\{A_{12} \pm \left(\frac{2c_2}{c_1^2} - 1\right)^{\frac{1}{2}}(A_{11}A_{22})^{\frac{1}{2}}\cos V\right\}, \end{aligned}\right\} \tag{5.17}$$

‡ The equivalence of $(5.8)_2$ and (5.10) follows also from Ricci's theorem $(2.9)_1$, as well as $(2.6)_1$ and (2.7).
§ Details of the integration process are given in Appendix A.

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where the parameters c_1, c_2 are restricted by $(4.11)_{1,2}$, $\bar{V}$ is defined by

$$\bar{V} = \int_{_0\theta^1}^{\theta^1} X_1(\xi, \theta^2)\, d\xi + \int_{_0\theta^2}^{\theta^2} X_2(_0\theta^1, \eta)\, d\eta + \phi_1, \tag{5.18}$$

$$\left.\begin{aligned} X_1 &= A^{-\frac{1}{2}}\left[-A_{11,2} + A_{12,1} + \tfrac{1}{2}A_{12}\left(\frac{A_{22,1}}{A_{22}} - \frac{A_{11,1}}{A_{11}}\right)\right], \\ X_2 &= A^{-\frac{1}{2}}\left[A_{22,1} - A_{12,2} + \tfrac{1}{2}A_{12}\left(\frac{A_{22,2}}{A_{22}} - \frac{A_{11,2}}{A_{11}}\right)\right], \end{aligned}\right\} \tag{5.19}$$

ϕ_1 is a constant and the integration in (5.18) is from a fixed material point $_0\theta^\alpha = \{_0\theta^1, _0\theta^2\}$ to a material point $\theta^\alpha = \{\theta^1, \theta^2\}$ in the two-dimensional region of space occupied by the surface $\mathscr{s}$.

Case (*II*): $\bar{K} \neq 0$. In this case, using the expanded form of $(4.8)_1$, from the condition (5.16) and the dual of $(2.4)_2$ we can easily show that the solution sought is given by (5.7), which states that the first fundamental forms of the initial surface $\mathscr{S}$ and the deformed surface $\mathscr{s}$ are proportional to each other. Hence, we have

LEMMA 5.1. *When $\bar{K} \neq 0$, the controllable deformations are so restricted that the initial surface $\mathscr{S}$ and the deformed surface $\mathscr{s}$ are in conformal (angle preserving) correspondence.*

Proof. The proof follows immediately from a theorem of differential geometry (see, e.g. theorem 36.1 of Eisenhart (1941)) and the property exhibited by the solution (5.7).

It may be observed that the class of deformation indicated in lemma 5.1 imply, for example, that a net of the coordinate curves which is initially orthogonal ($A_{12} = 0$) on the surface $\mathscr{S}$ will remain so on $\mathscr{s}$ after deformation.

Solution for $b_{\alpha\beta}$. Having obtained the above solutions for $a_{\alpha\beta}$, we now turn to the determination of the coefficients of the second fundamental form $b_{\alpha\beta}$ from the remaining equilibrium conditions $(4.8)_{5,6}$ and the compatibility requirements (2.18) and (2.19) in the deformed configuration. However, we first show that the condition $(4.8)_6$ is redundant when $\bar{K} \neq 0$. For this purpose, starting with $(4.8)_6$ and using (5.7) and $(4.8)_5$, it is easily seen that the parameter c_4 can be expressed in terms of c_1 and c_3 as

$$c_4 = \tfrac{1}{2}c_1 c_3, \tag{5.20}$$

rendering the condition $(4.8)_6$ redundant. Keeping this in mind, in obtaining the solution for $b_{\alpha\beta}$ we employ $(4.8)_5$, (2.18) and (2.19) for all values of $\bar{K}$, including $\bar{K} = 0$, and then check *a posteriori* if $(4.8)_6$ is identically satisfied (i.e. redundant) also upon specialization to $\bar{K} = 0$. A summary of the solution for $b_{\alpha\beta}$, classified according to certain range of values of $\bar{K}$, is given below but the details of the solution which are somewhat lengthy can be found in Appendix B.

(i) *For $\bar{K} < c_3^2/2c_1$*: In this case a solution for $b_{\alpha\beta}$ is given by

$$\left.\begin{aligned} \frac{b_{11}}{A_{11}} &= \tfrac{1}{2}\{c_3 \pm (c_3^2 - 2c_1\bar{K})^{\frac{1}{2}}(A_{11}A_{22})^{-\frac{1}{2}}[A_{12}\cos\bar{V} + A^{\frac{1}{2}}\sin\bar{V}]\}, \\ \frac{b_{22}}{A_{22}} &= \tfrac{1}{2}\{c_3 \pm (c_3^2 - 2c_1\bar{K})^{\frac{1}{2}}(A_{11}A_{22})^{-\frac{1}{2}}[A_{12}\cos\bar{V} - A^{\frac{1}{2}}\sin\bar{V}]\}, \\ b_{12} &= \tfrac{1}{2}\{c_3 A_{12} \pm (c_3^2 - 2c_1\bar{K})^{\frac{1}{2}}(A_{11}A_{22})^{\frac{1}{2}}\cos\bar{V}\}, \end{aligned}\right\} \tag{5.21}$$

provided $A_{\alpha\beta}$ and $\bar{K}$ also satisfy the integrability condition

$$\bar{X}_{1,2} = \bar{X}_{2,1}, \tag{5.22}$$

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where
$$\left.\begin{aligned}\bar{X}_1 &= A^{-\frac{1}{2}}\left[-A_{11,2}+A_{12,1}+\tfrac{1}{2}A_{12}\left(\frac{A_{22,1}}{A_{22}}-\frac{A_{11,1}}{A_{11}}\right)+\frac{c_1(A_{11}\bar{K}_{,2}-A_{12}\bar{K}_{,1})}{c_3^2-2c_1\bar{K}}\right],\\ \bar{X}_2 &= A^{-\frac{1}{2}}\left[A_{22,1}-A_{12,2}+\tfrac{1}{2}A_{12}\left(\frac{A_{22,2}}{A_{22}}-\frac{A_{11,2}}{A_{11}}\right)+\frac{c_1(A_{12}\bar{K}_{,2}-A_{22}\bar{K}_{,1})}{c_3^2-2c_1\bar{K}}\right]\end{aligned}\right\} \tag{5.23}$$

and where
$$\bar{V} = \int_{{}_0\theta^1}^{\theta^1}\bar{X}_1(\xi,\theta^2)\,\mathrm{d}\xi+\int_{{}_0\theta^2}^{\theta^2}\bar{X}_2({}_0\theta^1,\eta)\,\mathrm{d}\eta+\bar{\phi}_1. \tag{5.24}$$

After substituting from (5.23) and using (2.11), $(2.10)_{4,5}$, the dual of $(2.4)_2$ and $(2.21)_2$, we can reduce the condition (5.22) to

$$\begin{aligned}4A\bar{K}+\frac{c_1}{c_3^2-2c_1\bar{K}}&\left[\frac{A_{,1}}{A}(A_{12}\bar{K}_{,2}-A_{22}\bar{K}_{,1})+\frac{A_{,2}}{A}(A_{12}\bar{K}_{,1}-A_{11}\bar{K}_{,2})\right.\\ &\left.+2(A_{11}\bar{K}_{,2}-A_{12}\bar{K}_{,1})_{,2}+2(A_{22}\bar{K}_{,1}-A_{12}\bar{K}_{,2})_{,1}\right]\\ &+\frac{4c_1^2}{(c_3^2-2c_1\bar{K})^2}[A_{11}(\bar{K}_{,2})^2+A_{22}(\bar{K}_{,1})^2-2A_{12}\bar{K}_{,1}\bar{K}_{,2}] = 0.\end{aligned} \tag{5.25}$$

(ii) *For* $\bar{K} = c_3^2/2c_1 \geqslant 0$: In this case, a solution for $b_{\alpha\beta}$ is given by‡
$$b_{\alpha\beta} = \tfrac{1}{2}c_3A_{\alpha\beta} = \mp(\tfrac{1}{2}c_1)^{\frac{1}{2}}\bar{K}^{\frac{1}{2}}A_{\alpha\beta}. \tag{5.26}$$

(iii) *For* $\bar{K} > c_3^2/2c_1$: No solution exists in this case.

Next for later convenience, we specialize the general results stated in (i) and (ii) above to the case in which the Gaussian curvature $\bar{K}$ vanishes, i.e. the initial surface $\mathscr{S}$ is developable. In this case, if $c_3 \neq 0$, it is easily seen that the requirement $\bar{K} < c_3^2/2c_1$ is fulfilled and the criterion (5.25) is satisfied. The corresponding solution stated below follows from (5.21).

(iv) *For* $\bar{K} = 0$. In this case the solution for $b_{\alpha\beta}$ is given by§
$$\left.\begin{aligned}b_{11}/A_{11} &= \tfrac{1}{2}c_3\{1\pm(A_{11}A_{22})^{-\frac{1}{2}}[A_{12}\cos W+A^{\frac{1}{2}}\sin W]\},\\ b_{22}/A_{22} &= \tfrac{1}{2}c_3\{1\pm(A_{11}A_{22})^{-\frac{1}{2}}[A_{12}\cos W-A^{\frac{1}{2}}\sin W]\},\\ b_{12} &= \tfrac{1}{2}c_3\{A_{12}\pm(A_{11}A_{22})^{\frac{1}{2}}\cos W\},\end{aligned}\right\} \tag{5.27}$$

where
$$W = \bar{V}|_{\bar{K}=0} = \int_{{}_0\theta^1}^{\theta^1}X_1(\xi,\theta^2)\,\mathrm{d}\xi+\int_{{}_0\theta^2}^{\theta^2}X_2({}_0\theta^1,\eta)\,\mathrm{d}\eta+\bar{\phi}_1. \tag{5.28}$$

If, on the other hand, $c_3 = 0$ the requirement $\bar{K} = c_3^2/2c_1$ is satisfied and from (5.26) we have
$$b_{\alpha\beta} = 0 \quad (\bar{K} = 0, c_3 = 0). \tag{5.29}$$

It then follows that the solution (5.29) can be obtained also as a special case of (5.27) by setting $c_3 = 0$. Therefore, when $\bar{K} = 0$, we regard (5.27) as the solution of $b_{\alpha\beta}$ for all values of c_3.

With the use of the results (5.27) and (5.17), the deformation parameter c_4 in $(4.8)_6$ after a lengthy but straightforward manipulation can be expressed in terms of c_1, c_2, c_3 as
$$c_4 = \tfrac{1}{2}c_1c_3\left[1+(\pm)_a(\pm)_b\left(\frac{2c_2}{c_1^2}-1\right)^{\frac{1}{2}}\cos(\phi_1-\bar{\phi}_1)\right] \quad (\bar{K} = 0) \tag{5.30}$$

and this verifies that the condition $(4.8)_6$ is also redundant for $\bar{K} = 0$. The symbols $(\pm)_a$ and $(\pm)_b$ in (5.30) refer to the choice of signs in the expressions (5.17) and (5.27) for $a_{\alpha\beta}$ and $b_{\alpha\beta}$, respectively.

‡ The order of choice of sign on the right-hand side of (5.26) is for consistency in later developments.

§ The temporary notation W in (5.27) and (5.28) should not be confused with the use of the same symbol for a different purpose in section 3.

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Before proceeding further, we state the following two lemmas:

LEMMA 5.2. *If in a controllable deformation the Gaussian curvature of the initial surface* $\mathscr{S}$ *is a nonzero constant, then it must be positive* ($\bar{K} = c_3^2/2c_1 > 0$) *and* $\mathscr{S}$ *is a spherical surface.*‡

Proof. Recall first that the above solution for $b_{\alpha\beta}$ has been classified according to (i) $\bar{K} < c_3^2/2c_1$ and (ii) $\bar{K} = c_3^2/2c_1 \geqslant 0$. In case (i) if $\bar{K}$ is a nonzero constant, then the criterion (5.25) reduce to $4A\bar{K} = 0$ and (since $A \neq 0$) this leads to $\bar{K} = 0$ which is a contradiction. Hence, it is not possible to have a nonzero constant in case (i). Turning to case (ii) it is at once evident that if $\bar{K}$ is a nonzero constant, it must be positive. This completes the proof.

LEMMA 5.3. *When* $\bar{K}$ = *positive constant so that* $\mathscr{S}$ *is a spherical surface, the controllable deformations are so restricted that a net of orthogonal trajectories* ($A_{12} = 0$) *on* $\mathscr{S}$ *deform into a net of lines of curvature* ($a_{12} = b_{12} = 0$) *on* $\mathscr{s}$.

Proof. The proof follows immediately from (5.7) and (5.26). This result is certainly stronger than lemma 5.1 which holds for $\bar{K} \neq 0$.

We record now a summary of the results for $b_{\alpha\beta}$ in a manner which will be particularly useful in our subsequent developments:

$$\text{Solutions of } b_{\alpha\beta} \text{ are given by:} \left\{ \begin{array}{lll} (5.27) & \text{when} & \bar{K} = 0; \\ (5.26) & \text{when} & \bar{K} = c_3^2/2c_1 > 0; \\ (5.21) & \text{when} & \text{(i) } \bar{K} \neq \text{constant}, \\ & & \text{(ii) } \bar{K} < c_3^2/2c_1 \text{ and} \\ & & \text{(iii) if criterion (5.22) is satisfied;} \\ & & \quad \text{and no solution exists when} \\ & & \quad \bar{K} > c_3^2/2c_1. \end{array} \right\} \quad (5.31)$$

It should be emphasized that the above solutions for $b_{\alpha\beta}$ hold for all values of f^3 or the surface pressure p including $p = 0$. In the absence of the surface load ($f^3 = 0$), i.e. when the membrane is deformed by edge loads alone, the various expressions in the solutions (5.27), (5.26) and (5.21) simplify considerably and the appropriate results may be obtained from these solutions by setting $c_3 = 0$, in view of lemma 4.1. One noteworthy result, in this connection, may be stated as

LEMMA 5.4. *If the membrane can be controllably deformed by edge loads alone, then the initial surface must be either a developable surface* ($\bar{K} = 0$) *or a surface of negative Gaussian curvature* ($\bar{K} < 0$).

Proof. Recall from (5.31) the classification of solutions for $b_{\alpha\beta}$ (whenever these exist) according to (*a*) $\bar{K} = 0$, (*b*) $\bar{K} = c_3^2/2c_1 > 0$ and (*c*) $\bar{K} < c_3^2/2c_1$. When the membrane is controllably deformed by edge loads alone ($f^3 = 0$), the deformation parameter $c_3 = 0$ by virtue of lemma 4.1. Then, the condition (*b*) gives $\bar{K} = 0 > 0$ which is not possible and (*c*) yields $\bar{K} < 0$. Thus, the only possibilities for the initial Gaussian curvature are $\bar{K} = 0$ or $\bar{K} < 0$. This completes the proof.

Before closing this section, for later convenience we combine the solutions obtained in this section for $a_{\alpha\beta}$ and $b_{\alpha\beta}$. These solutions which involve the initial metric tensor $A_{\alpha\beta}$ and the

‡ The term spherical surface is used here in the sense of Eisenhart (1941, §49). Thus, a *spherical surface* is one whose Gaussian curvature is a positive constant. By contrast a sphere refers to a surface of constant radius, or more precisely, a *sphere* of radius r is the set of all points at the distance r from a fixed point called the centre of sphere (see also O'Neill 1966, p. 128).

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constant deformation parameters c_1, c_2, c_3 depend also on certain range of values of the initial Gaussian curvature $\bar{K}$ and may be summarized as follows:

$$\left.\begin{cases}(1)\ \text{The solutions (5.17) and (5.27) when } \bar{K}=0;\\ (2)\ \text{The solutions (5.7) and (5.26) when } \bar{K}=c_3^2/2c_1>0;\text{ and}\\ (3)\ \text{The solutions (5.7) and (5.21) when (i) } \bar{K}\neq\text{const., (ii) } \bar{K}<c_3^2/2c_1 \text{ and (iii) if}\\ \quad\ \text{criterion (5.22) is satisfied.}\end{cases}\right\} \tag{5.32}$$

6. Static controllable solutions

With the use of (5.7) and (4.10) the deformation parameter c_2 can be expressed in terms of c_1 alone. To see this, we note from (5.7) that $a_{11}/A_{11} = a_{22}/A_{22} = \frac{1}{2}c_1$, $a_{12} = (c_1/2)\,A_{12}$ and after substituting this into (4.10) we easily obtain the result $2c_2 - c_1^2 = 0$. It then follows that the invariants I_1 and I_2 in $(4.8)_{1,2}$ can be expressed in terms of c_1 only when $\bar{K}\neq 0$, i.e.

$$I_1 = c_1,\quad I_2 = \tfrac{1}{2}c_1^2 \quad (\bar{K}\neq 0, c_1>0). \tag{6.1}$$

Further, in anticipation of certain results to be recorded below, we note that when $\bar{K}\neq 0$ the functions Ψ_α in (2.32) and $\hat{\psi}$ in (2.29) may be regarded as different functions of c_1 alone. Hence, in view of (6.1),

$$\psi = \hat{\psi}(I_1, I_2) = \psi'(c_1) \quad (\bar{K}\neq 0), \tag{6.2}$$

$$\frac{d\psi'}{dc_1} = \frac{\partial\hat{\psi}}{\partial I_1}\frac{dI_1}{dc_1} + \frac{\partial\hat{\psi}}{\partial I_2}\frac{dI_2}{dc_1} = \Psi_1(I_1, I_2) + c_1\Psi_2(I_1, I_2). \tag{6.3}$$

For later reference, with the help of (2.24), (2.31), $(2.25)_2$ and (5.32), we also record below the expressions for the mass density ρ, the resultants $N^{\alpha\beta}$ and the assigned force f^3 (or the pressure p) appropriate for certain special cases which are of particular interest here:

(*a*) If the initial surface $\mathscr{S}$ is developable ($\bar{K}=0$), then the mass density ρ, the assigned force f^3 per unit mas and the components of $N^{\alpha\beta}$ are given by

$$\left.\begin{aligned}\rho &= \rho_0\left(\frac{2}{c_1^2-c_2}\right)^{\frac{1}{2}},\\ f^3 &= -2c_3[(\Psi_1+c_1\Psi_2) + (\pm)_a(\pm)_b\Psi_2(2c_2-c_1^2)^{\frac{1}{2}}\cos(\phi_1-\bar{\phi}_1)]\end{aligned}\right\} \tag{6.4}$$

and

$$\left.\begin{aligned}\frac{N^{11}}{A^{11}} &= 2\rho\Psi_1 + 2\rho c_1\Psi_2\left\{1\mp\left(\frac{2c_2}{c_1^2}-1\right)^{\frac{1}{2}}(A_{11}A_{22})^{-\frac{1}{2}}[A_{12}\cos V - A^{\frac{1}{2}}\sin V]\right\},\\ \frac{N^{22}}{A^{22}} &= 2\rho\Psi_1 + 2\rho c_1\Psi_2\left\{1\mp\left(\frac{2c_2}{c_1^2}-1\right)^{\frac{1}{2}}(A_{11}A_{22})^{-\frac{1}{2}}[A_{12}\cos V + A^{\frac{1}{2}}\sin V]\right\},\\ N^{12} &= 2\rho\Psi_1A^{12} + 2\rho c_1\Psi_2\left\{A^{12}\pm\left(\frac{2c_2}{c_1^2}-1\right)^{\frac{1}{2}}(A^{11}A^{22})^{\frac{1}{2}}\cos V\right\}.\end{aligned}\right\} \tag{6.5}$$

Also, the arguments I_γ ($\gamma = 1, 2$) of the functions Ψ_α in this case are given by $(4.8)_{1,2}$ subject to the restrictions $(4.11)_1$ and (4.13).

(*b*) If the initial surface $\mathscr{S}$ is a spherical surface isometric‡ to a surface of sphere of radius $1/\bar{K}^{\frac{1}{2}}$, the solution for ρ, f^3 and $N^{\alpha\beta}$ is

$$\left.\begin{aligned}\rho &= (2/c_1)\,\rho_0,\\ f^3 &= \mp 2(2c_1)^{\frac{1}{2}}\bar{K}^{\frac{1}{2}}(\Psi_1+c_1\Psi_2) = \mp 2(2c_1)^{\frac{1}{2}}\bar{K}^{\frac{1}{2}}\,d\psi'/dc_1 \neq 0\end{aligned}\right\} \tag{6.6}$$

‡ Recall that isometric surfaces are those whose first fundamental forms are identical after admissible coordinate transformations (see Eisenhart 1941, p. 146).

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and
$$N^{\alpha\beta} = 2\rho(\Psi_1 + c_1\Psi_2)\,A^{\alpha\beta} = 2\rho(\mathrm{d}\psi'/\mathrm{d}c_1)\,A^{\alpha\beta}, \tag{6.7}$$

where the arguments I_γ of the functions Ψ_α in (2.32) are given by (6.1) and where (6.3) has been used in deriving the second of (6.7) and $(6.6)_2$.

(*c*) If $\mathscr{S}$ is a surface of variable Gaussian curvature ($\bar{K} \neq$ const.), which satisfies the criterion (5.22), we have

$$\left.\begin{aligned} \rho &= (2/c_1)\,\rho_0, \\ f^3 &= -2c_3(\Psi_1 + c_1\Psi_2) = -2c_3\,\mathrm{d}\psi'/\mathrm{d}c_1 \end{aligned}\right\} \tag{6.8}$$

and
$$N^{\alpha\beta} = 2\rho(\Psi_1 + c_1\Psi_2)\,A^{\alpha\beta} = 2\rho(\mathrm{d}\psi'/\mathrm{d}c_1)\,A^{\alpha\beta}, \tag{6.9}$$

where the argument I_γ of the functions Ψ_α are again given by (6.1).

Let $\theta = \theta^\alpha(s)$, with s as the arc length, represent the parametric equations of a curve c on $\mathscr{s}$ which may be identified with a closed boundary curve of $\mathscr{s}$; and let $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ denote, respectively, the unit tangent and the unit normal to c. Then, recalling the expressions

$$\left.\begin{aligned} \boldsymbol{\lambda} &= \left(\frac{\partial \boldsymbol{r}}{\partial s}\cdot\frac{\partial \boldsymbol{r}}{\partial s}\right)^{-\frac{1}{2}}\frac{\partial \boldsymbol{r}}{\partial s} = \left(a_{\lambda\nu}\frac{\mathrm{d}\theta^\lambda}{\mathrm{d}s}\frac{\mathrm{d}\theta^\nu}{\mathrm{d}s}\right)^{-\frac{1}{2}}\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s}\,\boldsymbol{a}_\alpha, \\ \boldsymbol{\nu} &= \boldsymbol{a}_3\times\boldsymbol{\lambda} = \left(a_{\lambda\nu}\frac{\mathrm{d}\theta^\lambda}{\mathrm{d}s}\frac{\mathrm{d}\theta^\nu}{\mathrm{d}s}\right)^{-\frac{1}{2}}\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s}\,\epsilon_{\alpha\beta}\,\boldsymbol{a}^\beta, \end{aligned}\right\} \tag{6.10}$$

the edge loads which must be prescribed along the boundary curve $\theta^\alpha = \theta^\alpha(s)$ of $\mathscr{s}$ to maintain the controllable deformations can be calculated by means of $(2.23)_2$.

Preliminary to our main objective and for ease of reference, we recall several well-known theorems as follows:

THEOREM 6.1. *Surfaces with constant mean and Gaussian curvatures must be planes, right circular cylinders or spheres.*‡

THEOREM 6.2. (Weatherburn (1930), p. 194). *Every surface of positive constant Gaussian curvature K is isometric to a sphere of radius* $1/K^{\frac{1}{2}} > 0$.

THEOREM 6.3. (O'Neill (1966), p. 263). *The only closed (complete) surface with a positive constant Gaussian curvature is a sphere.*§

In consequence of the form of the solutions for $a_{\alpha\beta}$ and $b_{\alpha\beta}$ found in §5, with the help of (B 1) of Appendix B, $(2.17)_1$ and (2.19), we record the expressions for the mean and the Gaussian curvatures of the deformed surface $\mathscr{s}$ as

$$H = \text{const.} = \left\{\begin{matrix} \dfrac{c_1c_3 - c_4}{c_1^2 - c_2} & \text{if} \quad \bar{K} = 0, \\ c_3/c_1 & \text{if} \quad \bar{K} \neq 0, \end{matrix}\right\} \tag{6.11}$$

$$K = (2/c_1)\,\bar{K} \quad (c_1 > 0), \tag{6.12}$$

where $c_2 = \frac{1}{2}c_1^2$ for $\bar{K} \neq 0$ and the redundant parameter c_4 is given by (5.20) for $\bar{K} \neq 0$ and by (5.30) for $\bar{K} = 0$.

‡ The proof of this theorem can be effected by means of the Gauss equation (2.19) and the Mainardi–Codazzi equations (2.18). The theorem was stated and used by Ericksen (1954, p. 473). The terms cylinders and spheres refer to sectors of cylindrical surfaces and surfaces of spheres, respectively.

§ The term *complete* is used synonymously with closed. For surfaces in $\mathscr{E}^3$, the terms complete and closed are equivalent; see O'Neill (1966, p. 263).

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We now consider several useful lemmas which follow from the conclusions (6.11) and (6.12) and the theorems 6.1–6.3. These lemmas, which have significant implications in regard to controllable solutions sought, may be stated as follows:

LEMMA 6.1. *A sector of an initial undeformed surface can be deformed into either a plane or a sector of a right circular cylinder if and only if the initial surface is developable, i.e.* $\bar{K} = 0$.

Proof. If $\bar{K} = 0$, from (6.11) and (6.12), we have $H =$ const. and $K = 0$ and by theorem 6.1 the deformed surface is either a plane or a right circular cylinder. Conversely, if the deformed surface is either a plane or a right circular cylinder, then $K = 0$ and we see that $\bar{K} = 0$ also and the lemma is proved.

LEMMA 6.2. *A sector of an initial undeformed surface can be deformed into a sector of a sphere of radius r if and only if the initial surface is a spherical surface ($\bar{K}$ = constant > 0) isometric to a sector of sphere of radius* $R = r/(c_1/2)^{\frac{1}{2}} = (2c_1)^{\frac{1}{2}}/|c_3|$.

Proof. If the initial surface is a sector of a spherical surface, i.e. $\bar{K}$ = positive constant, then by (6.12) and (6.11), K = constant $\neq 0$ and H = constant. Hence, by theorem 6.1, the deformed surface is a sector of a sphere. Conversely if the deformed surface is a sphere, i.e., K = positive constant, then $\bar{K}$ = positive constant by (6.12) and it follows from theorem 6.2 that the initial surface is a spherical surface isometric to a sphere. The radius of the latter is obtained from $\bar{K} = 1/R^2 = c_3^2/2c_1$ and by (6.12) we have $1/r^2 = 2/(c_1 R^2)$ or $r = (\frac{1}{2}c_1)^{\frac{1}{2}}R$ and the results stated in the lemma follow.

LEMMA 6.3. *The only closed surface which can be controllably deformed into a closed (i.e. complete) sphere is a closed (i.e. complete) sphere.*

Proof. The proof follows immediately from lemma 6.2 and theorem 6.3.

LEMMA 6.4. *The specification $\boldsymbol{f} = \boldsymbol{0}$, i.e. the absence of surface load, implies that either the deformation parameter $c_3 = 0$ and/or the deformed surface is minimal ($H = 0$) and conversely*:

$$\boldsymbol{f} = \boldsymbol{0} \Leftrightarrow c_3 = 0 \Leftrightarrow H = 0. \tag{6.13}$$

Proof. It is clear from (5.20) and (5.30) that $c_4 = 0$ whenever $c_3 = 0$. It then follows from lemma 4.1 that $\boldsymbol{f} = \boldsymbol{0}$ if and only if $c_3 = 0$. The second part of (6.13), namely that $c_3 = 0$ if and only if $H = 0$, follows at once from (6.11).

So far we have shown the existence of controllable solutions for three mutually exclusive categories of initial surfaces, namely (1) developable surfaces ($\bar{K} = 0$), (2) spherical surfaces ($\bar{K}$ = positive constant) and (3) surfaces with nonconstant Gaussian curvature

$$\bar{K} < c_3^2/2c_1. \tag{6.14}$$

In the last category the initial surface must be further restricted in that its $A_{\alpha\beta}$ and $\bar{K}$ must also satisfy the differential criterion (5.22). Keeping the latter in mind, it is natural to inquire as to the nature of the additional restrictions (if any) imposed by (5.22). More specifically, for given deformation parameters c_1 and c_3, we may ask the following two questions: (*a*) Do initial surfaces exist whose metric tensor $A_{\alpha\beta}$ and Gaussian curvature $\bar{K}$ satisfy (5.22), in addition to the initial compatibility equations $(2.21)_{1,2}$? (*b*) If the answer to (*a*) is in the affirmative, is the criterion (5.22) satisfied by every surface of nonconstant Gaussian curvature in $\mathscr{E}^3$? In order to answer the first question, we show that consideration of the differential criterion (5.22) does not lead to nonexistence and conclude, therefore, that initial surfaces do exist whose $A_{\alpha\beta}$ and $\bar{K}$ satisfy

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(5.22). With reference to the second question, we observe that if the $A_{\alpha\beta}$ and $\bar{K}$ of all initial surfaces in $\mathscr{E}^3$ (compatible with controllable deformations) satisfy the criterion (5.22) identically then clearly (5.22) places no additional restriction on $A_{\alpha\beta}$ and $\bar{K}$; but, in fact, we show the contrary and exhibit a further restriction imposed by (5.22) on the geometrical properties of the initial surface.

For the purpose of providing the answers to the above questions, it will suffice to consider a subclass of all possible initial surfaces in $\mathscr{E}^3$ consistent with the condition (6.14). In fact, for convenience, we take this subclass of initial surfaces to be those with constant mean and negative nonconstant Gaussian curvatures. We first demonstrate below that such surfaces exist without integration of (5.22) and hence answer the first question. We further show that the initial constant mean curvature cannot be prescribed arbitrarily and hence provide the answer to the second question, which also implies the criterion (5.22) is not satisfied by every initial surface in $\mathscr{E}^3$.

Thus, let $\mathscr{S}$ be an initial surface whose mean and Gaussian curvatures are specified by

$$\bar{H} = \eta, \quad \bar{K} < 0, \tag{6.15}$$

where η is an arbitrary prescribed constant.‡ Also, in order to simplify the analysis that follows, let the convected coordinates θ^α coincide with a net of lines of curvature coordinates on $\mathscr{S}$. Then, $A_{12} = B_{12} = 0$ and the coefficients of the first and second fundamental forms are

$$A_{\alpha\beta} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B_{\alpha\beta} = \begin{pmatrix} A_{11}/R_1 & 0 \\ 0 & A_{22}/R_2 \end{pmatrix}, \tag{6.16}$$

where R_1 and R_2 are the principal radii of curvature on $\mathscr{S}$. With the use of $(6.16)_1$ and (5.23), the criterion (5.22) in lines of curvature coordinates reduces to

$$\left\{\frac{1}{A^{\frac{1}{2}}}\left[A_{11,2} - \frac{c_1 A_{11} \bar{K}_{,2}}{c_3^2 - 2c_1 \bar{K}}\right]\right\}_{,2} + \left\{\frac{1}{A^{\frac{1}{2}}}\left[A_{22,1} - \frac{c_1 A_{22} \bar{K}_{,1}}{c_3^2 - 2c_1 \bar{K}}\right]\right\}_{,1} = 0. \tag{6.17}$$

We also need to express the initial compatibility equations $(2.21)_{1,2}$ in terms of the principal radii of curvature of $\mathscr{S}$. Thus, using (6.16), $(2.10)_{4,5}$, (2.11), the duals of $(2.4)_2$, $(2.6)_1$, $(2.8)_3$ and $(2.17)_2$, we write $(2.21)_1$ and $(2.21)_2$ as

$$\left.\begin{aligned} \left(\frac{1}{R_1}\right)_{,2} + \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)(\ln A_{11})_{,2} = 0, \\ \left(\frac{1}{R_2}\right)_{,1} + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)(\ln A_{22})_{,1} = 0 \end{aligned}\right\} \tag{6.18}$$

and

$$\frac{1}{R_1 R_2} = -\frac{1}{2A^{\frac{1}{2}}}\left[\left(\frac{A_{22,1}}{A^{\frac{1}{2}}}\right)_{,1} + \left(\frac{A_{11,2}}{A^{\frac{1}{2}}}\right)_{,2}\right], \tag{6.19}$$

respectively. By the definitions of the mean and Gaussian curvatures of $\mathscr{S}$, i.e., the duals of (2.17), we have

$$1/R_1 = \bar{H} + (\bar{H}^2 - \bar{K})^{\frac{1}{2}}, \quad 1/R_2 = \bar{H} - (\bar{H}^2 - \bar{K})^{\frac{1}{2}}, \tag{6.20}$$

where

$$\bar{H}^2 - \bar{K} \geqslant 0 \tag{6.21}$$

and the nonnegative restriction in (6.21) is required if the solutions (6.20) for $1/R_1$ and $1/R_2$ are to be real.

‡ The constant η in (6.15) need not be confused with the temporary use of the same symbol for a different purpose in §5.

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Before proceeding further, we need to examine if R_1 and R_2 as given by (6.20) are differentiable everywhere or not. It is easily seen that these functions are differentiable everywhere except at umbilics, i.e. at points where $R_1 = R_2$ or equivalently when $\bar{H}^2 - \bar{K} = 0$. Thus, we restrict our attention to initial surfaces which are nowhere umbilic, i.e. to initial surfaces whose constant mean and nonconstant negative Gaussian curvatures satisfy the condition

$$\bar{H}^2 - \bar{K} > 0. \tag{6.22}$$

It is evident that the requirements $(6.15)_{1,2}$ automatically fulfill (6.22); in fact, our choice for the initial surfaces of constant mean and negative Gaussian curvatures was motivated in anticipation of the requirement (6.22). However, as will become apparent below, the restriction (6.22) will not be needed if either the initial surface $\mathscr{S}$ or the deformed surface s is a nondevelopable minimal surface, i.e.

$$\left.\begin{array}{ll}(1)\ \text{either}\ \bar{H} = 0, & \bar{K} \neq 0,\\ (2)\quad \text{or}\quad H = 0, & K \neq 0.\end{array}\right\} \tag{6.23}$$

In the case (1) of (6.23), it follows from (6.21) that $\bar{K} < 0$. In the case (2) of (6.23), on the other hand, from the dual of (6.21) we have $K < 0$ and recalling (6.12) we again conclude that $\bar{K} = \frac{1}{2}c_1 K < 0$.

Returning to the consideration of the initial surfaces specified by (6.15), we substitute (6.20) into (6.18) and (6.19) and obtain the initial compatibility equations in the forms

$$\left.\begin{array}{l}\bar{H}_{,2} + \frac{1}{2}(\bar{H}^2 - \bar{K})^{-\frac{1}{2}}(2\bar{H}\bar{H}_{,2} - \bar{K}_{,2}) + (\bar{H}^2 - \bar{K})^{\frac{1}{2}}(\ln A_{11})_{,2} = 0,\\ \bar{H}_{,1} - \frac{1}{2}(\bar{H}^2 - \bar{K})^{-\frac{1}{2}}(2\bar{H}\bar{H}_{,1} - \bar{K}_{,1}) - (\bar{H}^2 - \bar{K})^{\frac{1}{2}}(\ln A_{22})_{,1} = 0\end{array}\right\} \tag{6.24}$$

and

$$-\bar{K} = \frac{1}{2A^{\frac{1}{2}}}\left[\left(\frac{A_{22,1}}{A^{\frac{1}{2}}}\right)_{,1} + \left(\frac{A_{11,2}}{A^{\frac{1}{2}}}\right)_{,2}\right]. \tag{6.25}$$

We now proceed to obtain the restriction demanded by the differential criterion (6.17) when the initial surface is specified by (6.15). Since the initial mean curvature $\bar{H}$ is constant, $\bar{H}_{,\alpha} = 0$. By using this result, the compatibility equations (6.24) reduce to

$$A_{11}\bar{K}_{,2} = 2A_{11,2}(\eta^2 - \bar{K}), \quad A_{22}\bar{K}_{,1} = 2A_{22,1}(\eta^2 - \bar{K}), \quad (\eta^2 - \bar{K} > 0). \tag{6.26}$$

Introducing the expressions $(6.26)_{1,2}$ into (6.17) and making use of (6.25) wherever the expression for $(-\bar{K})$ occurs, we obtain

$$\left[\eta^2 - \tfrac{1}{2}c_1\left(\frac{c_3}{c_1}\right)^2\right]\left[\frac{(-\bar{K})A^{\frac{1}{2}}}{c_3^2 - 2c_1\bar{K}} + \frac{2c_1(\eta^2 - \bar{K})}{A^{\frac{1}{2}}(c_3^2 - 2c_1\bar{K})^2}\left\{\frac{(A_{11,2})^2}{A_{11}} + \frac{(A_{22,1})^2}{A_{22}}\right\}\right] = 0. \tag{6.27}$$

In view of $(6.15)_2$, $(2.10)_3$, (6.14), $(4.11)_1$, $(6.26)_3$ and $(4.9)_2$, it is easily seen that each of the terms $(-\bar{K})$, $A^{\frac{1}{2}}$, $(c_3^2 - 2c_1\bar{K})$, c_1, $(\eta^2 - \bar{K})$ and A_{11}, A_{22} is positive and it follows that the expression in the second square bracket of (6.27) is positive. Hence, we must have:

$$\eta^2 - \tfrac{1}{2}c_1(c_3/c_1)^2 = 0. \tag{6.28}$$

Substituting for η and c_3/c_1 from $(6.15)_1$ and $(6.11)_2$, we finally obtain the relation

$$\bar{H} = \pm(\tfrac{1}{2}c_1)^{\frac{1}{2}}H, \tag{6.29}$$

which is a restriction on the initial mean curvature $\bar{H}$ in a controllable deformation.

The foregoing development between (6.14) and (6.29) has been obtained for the subclass of initial surfaces specified by (6.15). In view of (6.29), it is clear that the differential criterion

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(6.17) – which is deduced from (5.22) – cannot be satisfied identically and does not lead to nonexistence of solution for $\bar{K} \neq$ constant. This provides the answer to the first question raised earlier [following (6.14)]. It then follows that (5.22) cannot be satisfied identically by every surface in $\mathscr{E}^3$ and this provides the answer to the second question. The results concerning the two questions posed earlier may be summarized by the following.‡

THEOREM 6.4. *In a controllable deformation sustained either by edge loads alone* ($f^3 = 0$) *or by loads which include a uniform normal pressure* ($f^3 \neq 0$), *there exists some initial surfaces whose metric coefficients and nonconstant Gaussian curvature satisfy the differential criterion* (5.22).

With the help of the solutions for $a_{\alpha\beta}$ and $b_{\alpha\beta}$ obtained in section 5 and the various results of this section, we are now in a position to turn to our main objective and deduce the main conclusions on the controllable solutions possible in every isotropic elastic membrane whose strain energy function is characterized by (2.29); the membrane or a sector of the membrane may be subjected to edge loads, as well as a uniform normal pressure measured either per unit mass or per unit area. Our main conclusions are listed below as theorems 6.5–6.7 corresponding, respectively, to three mutually exclusive cases in which $\bar{K} = 0$, $\bar{K} =$ positive constant and $\bar{K} \neq$ constant as listed also in (5.32).

THEOREM 6.5. *A sector of an initial surface $\mathscr{S}$ can be deformed either into a plane in the absence of surface loads* ($\boldsymbol{f} = \boldsymbol{0}$) *or into a sector of a right circular cylindrical surface under loads which include a uniform normal pressure* ($f^3 \neq 0$) *if and only if the initial surface $\mathscr{S}$ is a developable surface. In this case, the controllable solution is given by* (5.17), (5.27), (6.4) *and* (6.5).

THEOREM 6.6. *A sector of an initial surface $\mathscr{S}$ can be deformed into a sector of a sphere of radius r if and only if the initial surface is a spherical surface isometric to a sphere of radius* $R = 1/\bar{K}^{\frac{1}{2}} = r/(\frac{1}{2}c_1)^{\frac{1}{2}}$. *The deformation, in this case, is sustained by application of loads which include a nonzero uniform normal pressure and the net of orthogonal trajectories on $\mathscr{S}$ deforms into a net of lines of curvature on s. The controllable solution is given by* (5.7), (5.26), (6.6) *and* (6.7).

THEOREM 6.7. *In the presence (or absence) of surface load prescribed by a uniform normal pressure, in addition to edge loads, a sector of an initial surface $\mathscr{S}$ can be deformed into a sector of noncircular right cylindrical surface with a nonzero constant mean curvature (or a nondevelopable minimal surface with a negative Gaussian curvature) if and only if the initial surface is a nondevelopable surface (or a surface of negative Gaussian curvature) with $A_{\alpha\beta}$ and $\bar{K}$ satisfying the differential criterion* (5.22). *The resulting deformation in both cases, i.e. in the presence or absence of the surface load, is in conformal correspondence between $\mathscr{S}$ and s and the controllable solutions are given by* (5.7), (5.21), (6.8) *and* (6.9).

We close this section with several corollaries which follow from the above theorems:

COROLLARY 6.1. *Among all nondevelopable surfaces of constant mean curvature, only a nondevelopable minimal surface is a possible initial surface in a controllable deformation maintained by edge loads alone.*

Proof. In the absence of surface loads, $H = 0$ by lemma 6.4. It is then seen immediately from (6.29) that $\bar{H} = 0$ also and the corollary is proved.

COROLLARY 6.2. *Among all surfaces of nonzero constant mean curvature $\bar{H}$ and negative Gaussian curvature $\bar{K}$, only those whose curvatures are specified by* $\bar{H} = \pm(\frac{1}{2}c_1)^{\frac{1}{2}}H$, $\bar{K} = \frac{1}{2}c_1 K$ *are possible initial*

‡ A number of corollaries may be stated as a consequence of theorem 6.4, but we postpone recording these until later in this section.

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surfaces in a controllable deformation maintained by loads which may include a uniform normal pressure ($f^3 \neq 0$).

Proof. The proof follows at once from (6.29) and (6.12).

Corollary 6.3. *A nondevelopable minimal initial surface cannot sustain a nonzero uniform normal pressure. However, it can be controllably deformed into another minimal surface under the action of edge loads alone.*

Proof. Suppose that the initial minimal surface can be controllably deformed in the presence of a nonzero uniform normal pressure ($f^3 \neq 0$). Since the initial surface is minimal, $\bar{H} = 0$ and by (6.29) we also have $H = 0$. But, by lemma 6.4, the latter is equivalent to $f^3 = 0$. This leads to contradiction and the first part of the corollary is proved. On the other hand, under the action of edge loads alone, we have shown that $H = 0$ when $f^3 = 0$ and this establishes the second part of the corollary.

Corollary 6.4. *The only complete (i.e. closed) surface which can be controllably deformed into a complete sphere is a complete (closed) sphere. The deformation, in this case, is sustained by application of a nonzero uniform normal pressure alone.*

Proof. The proof follows immediately from theorems 6.6 and 6.3.

7. Some elastic membranes with variable Gaussian curvatures in the undeformed state

We consider in this section some explicit results for three cases of controllably deformed elastic membranes with the following specifications:

(A) Only the initial surface is in the form of nondevelopable surface of revolution.

(B) Both the initial undeformed and the deformed surfaces are in the forms of nondevelopable surfaces of revolution.

(C) Only the deformed surface is in the form of nondevelopable surface of revolution.

Although in the case (C) only the deformed surface is specified, it should be clear from (6.12) that even in this case the initial surface must have a variable Gaussian curvature. Moreover, it should be noted that the deformed surface in the case (A) and the initial undeformed surface in the case (C) are not necessarily surfaces of revolution.

In each of the above three cases, we introduce suitable lines of curvature coordinates on both initial and deformed surfaces and make use of the general theorems of § 6 to obtain the desired controllable solutions. In the two cases (A) and (C) the explicit results are obtained without integrating the differential criterion (5.22) but in one case, namely (B), (5.22) is actually integrated.

(A) *A membrane with its initial configuration in the form of a surface of revolution*

Let the initial surface $\mathscr{S}$ be generated by the rotation of a plane curve C through an angle $\Gamma \leqq 2\pi$ about an axis in its plane. This curve, called a meridian curve, has no double points and does not meet the axis of revolution except possibly at one or both of its end points. Let (R, Θ, Z) be a fixed system of cylindrical polar coordinates with the Z-axis being coincident with the axis of revolution. Then, the meridian curve C can be parameterized by

$$\left.\begin{aligned} R = R(S), \quad Z = Z(S), \\ (R')^2+(Z')^2 = 1, \quad (\ )' \equiv \mathrm{d}(\ )/\mathrm{d}S, \end{aligned}\right\} \tag{7.1}$$

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where the parameter S chosen as the arclength measured from a fixed point along C is specified by $(7.1)_3$. It is not difficult to see that the meridian curves (i.e. the S-curves) and the parallels (i.e. the Θ-curves) form a net of lines of curvature coordinates on $\mathscr{S}$. We introduce now the surface coordinates $Y^\alpha = (S, \Theta)$ on $\mathscr{S}$, and denote by $\bar{\boldsymbol{A}}_\alpha$ and $\bar{A}_{\alpha\beta}$ the base vectors and the metric tensor associated with the Y^α-coordinates. Further, we select the values of Y^α of a typical material point in the initial undeformed configuration to be the convected coordinates at that point so that

$$\theta^\alpha = Y^\alpha = (S, \Theta) \text{ on } \mathscr{S}. \tag{7.2}$$

With this choice, $\bar{\boldsymbol{A}}_\alpha = \boldsymbol{A}_\alpha$, $\bar{A}_{\alpha\beta} = A_{\alpha\beta}$, and the initial metric tensor $A_{\alpha\beta}$ and the initial Gaussian curvature are‡

$$A_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}, \quad \bar{K} = -R''/R \quad (R > 0). \tag{7.3}$$

The differential criterion (5.22), or its alternative (6.17), now reduces to

$$\tfrac{1}{2}R\left\{\ln\left[R^4\left(c_3^2 + 2c_1\frac{R''}{R}\right)\right]\right\}' = \gamma, \quad R'' > -\frac{c_3^2}{2c_1}R, \tag{7.4}$$

where γ is an integration constant. The restriction indicated by the inequality $(7.4)_2$ follows from $(7.3)_2$ and the condition $\bar{K} < c_3^2/2c_1$ in the solution (5.21) [see also (3) in (5.32)]. With the help of $(7.3)_{1,2}$ and (7.4) the solutions (5.7) and (5.21) yield

$$a_{\alpha\beta} = \tfrac{1}{2}c_1\begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix} \quad (R > 0) \tag{7.5}$$

and

$$\left.\begin{aligned} b_{11} &= \tfrac{1}{2}\{c_3 \pm (c_3^2 + 2c_1 R''/R)^{\frac{1}{2}} \sin \bar{V}\}, \\ b_{22} &= \tfrac{1}{2}R^2\{c_3 \mp (c_3^2 + 2c_1 R''/R)^{\frac{1}{2}} \sin \bar{V}\}, \\ b_{12} &= \pm \tfrac{1}{2}R(c_3^2 + 2c_1 R''/R)^{\frac{1}{2}} \cos \bar{V}, \end{aligned}\right\} \tag{7.6}$$

where now

$$\bar{V} = (\bar{\phi}_1 - \gamma\Theta_0) + \gamma\Theta \tag{7.7}$$

and Θ_0 is a fixed value along the parallels Θ with the range $0 \leqslant \Theta \leqslant \Gamma \leqslant 2\pi$.

Since the absence of surface load is equivalent to setting $c_3 = 0$ by lemma 6.4, the criterion (7.4) can be further reduced to

$$\tfrac{1}{2}R[\ln(R^3R'')]' = \gamma \quad (R'' > 0). \tag{7.8}$$

Moreover, in this case the coefficients of the first fundamental form are still given by (7.5), but the expressions in (7.6) simplify and read as

$$\left.\begin{aligned} b_{11} = \pm(c_1 R''/2R)^{\frac{1}{2}} \sin \bar{V}, \quad b_{22} = \mp R^2(c_1 R''/2R)^{\frac{1}{2}} \sin \bar{V}, \\ b_{12} = \pm R(c_1 R''/2R)^{\frac{1}{2}} \cos \bar{V}. \end{aligned}\right\} \tag{7.9}$$

The condition $(7.4)_1$ is a restriction for a sector of surface of revolution of variable Gaussian curvature and its right-hand side represents an arbitrary constant of integration. For a complete membrane of revolution, the surface $\mathscr{S}$ is complete, i.e. $\Gamma = 2\pi$, and the constant γ must then be an integer. The latter can be effected by imposing on (7.6) and (7.7) the periodicity condition

$$\boldsymbol{R}(S, 2\pi) = \boldsymbol{R}(S, 0), \quad \boldsymbol{r}(S, 2\pi) = \boldsymbol{r}(S, 0), \tag{7.10}$$

according to which the position vectors of any point of the 2π-meridian on $\mathscr{S}$ coincide with those of the zero-meridian.

‡ In order to ensure the positive definiteness of $A_{\alpha\beta}$, we require that $R > 0$ throughout $\mathscr{S}$. This excludes our consideration of what may happen at the poles of the surface of revolution at which points $R = 0$.

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The foregoing results can be summarized by the following two corollaries to theorems 6.5–6.7:

COROLLARY 7.1. *Among all nondevelopable surfaces of revolution, only those generated by the meridian curves specified by* (7.1) *and restricted by* (7.8) *are possible initial surfaces in a controllable deformation maintained by edge loads alone. Moreover, the controllable solutions result in deformed minimal surfaces characterized by* (7.5) *and* (7.9).

COROLLARY 7.2. *Among all nondevelopable surfaces of revolution, only those generated by the meridian curves specified by* (7.1) *and restricted by* (7.4) *are possible initial surfaces in a controllable deformation maintained by loads which include a uniform normal pressure* ($f^3 \neq 0$). *Moreover, the controllable solutions result in deformed surfaces of constant mean curvature characterized by* (7.5) *and* (7.6) *with the deformation parameters* c_1 (> 0) *and* c_3 *restricted by* $(7.4)_2$.

The results contained in the corollaries 7.1 and 7.2 are valid for initial surfaces of revolution which may be either (i) a sector of nondevelopable surfaces of revolution or (ii) a complete nondevelopable surface of revolution. These results, therefore, appear to slightly enlarge the previous controllable solutions for elastic membranes of revolution in which both the initial and the deformed surfaces are taken to be complete surfaces of revolution (see Adkins & Rivlin (1952) and Green & Adkins (1970), ch. 4).

(B) *A membrane with both its initial and deformed state in the forms of surfaces of revolution.*

We now make an additional assumption and as in Green & Adkins (1970, ch. 4) we suppose that the deformed configuration of the membrane represented by $\mathscr{s}$ is also in the form of a surface of revolution. Let (r, θ, z) be a system of cylindrical polar coordinates referred to the same cylindrical polar reference frame as (R, Θ, Z), and let s denote the arclength measured along the meridian curve c on $\mathscr{s}$, which was initially the curve C on $\mathscr{S}$. Then, c can be parameterized as

$$\left.\begin{aligned} r = r(s), \quad z = z(s), \\ (r^{\cdot})^2 + (z^{\cdot})^2 = 1, \quad (\;)^{\cdot} \equiv \frac{\mathrm{d}(\;)}{\mathrm{d}s}. \end{aligned}\right\} \tag{7.11}$$

Also, as in Green & Adkins (1970, ch. 4), we assume that the deformed surface $\mathscr{s}$ has the same axis of revolution as‡ $\mathscr{S}$, and that for each Θ the meridian curves on $\mathscr{S}$ are deformed into corresponding meridians on $\mathscr{s}$, so that§

$$\theta = \Theta. \tag{7.12}$$

We introduce now the surface coordinates

$$y^{\alpha} = (s, \theta) \quad \text{on} \quad \mathscr{s}, \tag{7.13}$$

and consider the deformation described by the transformation

$$y^{\alpha} = y^{\alpha}(Y^{\beta}), \quad \det[\partial y^{\alpha}/\partial Y^{\beta}] \neq 0 \tag{7.14}$$

or equivalently

$$y^{\alpha} = y^{\alpha}(\theta^{\beta}), \quad \det[\partial y^{\alpha}/\partial \theta^{\beta}] \neq 0, \tag{7.15}$$

‡ This implies that the z-axis is taken to be coincident with the Z-axis, since earlier in this section the Z-axis was taken to be coincident with the axis of revolution of $\mathscr{S}$.

§ Such an assumption could be introduced even if $\mathscr{S}$ and $\mathscr{s}$ are not complete surfaces of revolution.

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in view of $(7.2)_1$. Let $\bar{\boldsymbol{a}}_\alpha$ denote the base vectors associated with the y^α-coordinates on $\mathscr{s}$. Then, under the transformations (7.15), we have

$$\bar{\boldsymbol{a}}_\alpha = \frac{\partial \theta^\beta}{\partial y^\alpha}\boldsymbol{a}_\beta, \quad \boldsymbol{a}_\alpha = \frac{\partial y^\beta}{\partial \theta^\alpha}\bar{\boldsymbol{a}}_\beta. \tag{7.16}$$

Further, it follows from $(2.2)_1$, (2.3) and (7.16) that the unit normal $\bar{\boldsymbol{a}}_3$ associated with the y^α-coordinates is related to $\boldsymbol{a}_3$ by‡

$$\boldsymbol{a}_3 = \mu\bar{\boldsymbol{a}}_3, \quad \mu = \frac{|\det(\partial y^\alpha/\partial\theta^\beta)|}{\det(\partial y^\alpha/\partial\theta^\beta)} = \pm 1. \tag{7.17}$$

Also, let $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$ stand, respectively, for the coefficients of the first and the second fundamental forms of $\mathscr{s}$ associated with the y^α-coordinates. Then, by (7.11) and (7.13),

$$\bar{a}_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \bar{b}_{\alpha\beta} = \begin{pmatrix} -\dfrac{r^{\cdot\cdot}}{z^{\cdot}} & 0 \\ 0 & z^{\cdot} \end{pmatrix} \quad (r > 0) \tag{7.18}$$

and we also note the transformations

$$a_{\alpha\beta} = \frac{\partial y^\gamma}{\partial\theta^\alpha}\frac{\partial y^\delta}{\partial\theta^\beta}\bar{a}_{\gamma\delta}, \quad b_{\alpha\beta} = \mu\frac{\partial y^\gamma}{\partial\theta^\alpha}\frac{\partial y^\delta}{\partial\theta^\beta}\bar{b}_{\gamma\delta} \tag{7.19}$$

between $a_{\alpha\beta}$ and $\bar{a}_{\alpha\beta}$ and between $b_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$.

We now restrict attention to the special case in which $\mathscr{s}$ is a complete surface of revolution. With the help of the transformations $(7.15)_1$, the assumption (7.12) and use of (7.5), (7.6) and (7.18), comparison of the left-hand and the right-hand sides of (7.19) results in

$$\left.\begin{aligned} &\frac{\partial s}{\partial\Theta} = 0, \quad \frac{\partial s}{\partial S} = \frac{\mathrm{d}s}{\mathrm{d}S} = \left(\frac{c_1}{2}\right)^{\frac{1}{2}}, \quad \frac{r}{R} = \left(\frac{c_1}{2}\right)^{\frac{1}{2}}, \\ &\bar{\phi}_1 - \gamma\Theta_0 + \gamma\Theta = 2n\pi \pm \tfrac{1}{2}\pi \quad (n = 0, \pm 1, \pm 2, \ldots), \\ &c_3 \pm \left(c_3^2 + 2c_1\frac{R''}{R}\right)^{\frac{1}{2}} = -\mu(2c_1)^{\frac{1}{2}}\frac{R''}{[1-(R')^2]^{\frac{1}{2}}}, \\ &c_3 \mp \left(c_3^2 + 2c_1\frac{R''}{R}\right)^{\frac{1}{2}} = \mu(2c_1)^{\frac{1}{2}}\frac{[1-(R')^2]^{\frac{1}{2}}}{R}, \end{aligned}\right\} \tag{7.20}$$

where the arclengths s and S have been assumed to increase together and this enables us to take positive square roots of the expressions from which $(7.20)_{2,3}$ are obtained. Since the right-hand side of $(7.20)_4$ has a fixed value for each n and since $(7.20)_4$ must hold for all Θ, it follows that under assumption (7.12) the coefficient of Θ in $(7.24)_4$ must vanish. Hence, if $\mathscr{s}$ is a complete surface of revolution, we must have

$$\gamma = 0, \quad \bar{\phi}_1 = 2n\pi \pm \tfrac{1}{2}\pi \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{7.21}$$

By the definitions of the principal extension ratios λ_α, namely

$$\lambda_1 = \mathrm{d}s/\mathrm{d}S, \quad \lambda_2 = r/R, \tag{7.22}$$

it is clear that λ_α are constants throughout $\mathscr{s}$ in view of§ $(7.20)_{2,3}$. The explicit values of λ_α can be readily calculated with the use of $(7.20)_{2,3}$ but will not be recorded here.

‡ The possibility for a different choice of signs in $(7.17)_2$ is introduced for later convenience and in connection with the examples discussed in §8.

§ The fact that the principal extension ratios are constants is introduced as an additional assumption in existing solutions; see Green & Adkins (1970, ch. 4).

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In view of $(7.21)_1$, the differential criterion (7.4) can be integrated to yield

$$2c_1 R'' + c_3^2 R - \eta_1 R^{-3} = 0, \tag{7.23}$$

where η_1 is an arbitrary constant of integration. Next, recalling $(7.1)_{1,4}$, we write

$$R'' = \mathrm{d}^2R/\mathrm{d}S^2 = \mathrm{d}R'/\mathrm{d}S = \tfrac{1}{2}(\mathrm{d}/\mathrm{d}R)\,(R')^2, \quad R' = \mathrm{d}R/\mathrm{d}S. \tag{7.24}$$

Using the last result, from (7.23) we calculate an expression for $(R')^2$ and by taking its square root we obtain

$$\frac{\mathrm{d}(R^2)}{\{-(1/2c_1)\,[\eta_1 - 2\eta_2 R^2 + c_3^2 R^4]\}^{\frac{1}{2}}} = 2\,\mathrm{d}S, \tag{7.25}$$

where η_2 is an arbitrary constant of integration and where R and S are assumed to increase together. When the surface load $f^3 \neq 0$, or equivalently $c_3 \neq 0$ by lemma 4.1, a further integration of (7.25) yields

$$R^2 = \frac{\eta_2}{c_3^2} + \frac{1}{c_3^2}(\eta_2 - \eta_1 c_3^2)^{\frac{1}{2}} \sin \frac{c_3}{(2c_1)^{\frac{1}{2}}}(2S + \eta_3), \tag{7.26}$$

where η_3 is another constant of integration. It is clear from (7.26) that in order for R to be real, we must require

$$\eta_2^2 - \eta_1 c_3^2 > 0. \tag{7.27}$$

With the help of $(7.20)_{2,3}$, (7.26) provides an expression for r in the form

$$r^2 = L_2 + L_1 \sin(2Es + \alpha), \tag{7.28}$$

where $\quad L_1 = (c_1/2c_3^2)\,(\eta_2^2 - \eta_1 c_3^2)^{\frac{1}{2}}, \quad L_2 = (c_1/2c_3^2)\,\eta_2, \quad E = c_3/c_1, \quad \alpha = (c_3/(2c_1)^{\frac{1}{2}})\,\eta_3. \qquad (7.29)$

The mean curvature of the surface $\mathscr{s}$ can now be calculated from $(2.17)_1$, (7.18), $(7.11)_3$ and (7.28). Thus

$$H = \pm\tfrac{1}{2}[1 + 2E^2 L_1 \sin(2Es + \alpha)]\,[L_2 + L_1 \sin(2Es + \alpha) - E^2 L_1^2 \cos^2(2Es + \alpha)]^{\frac{1}{2}}. \tag{7.30}$$

Since both (7.30) and $(6.11)_2$ must hold simultaneously, by equating them and after using $(7.29)_{1,2,3}$ we have

$$\eta_2^2 - \eta_1 c_3^2 = 2\eta_1 c_1 - c_1^2. \tag{7.31}$$

With the use of (7.31) and $(7.29)_3$, the constants L_1 and L_2 in $(7.29)_{1,2}$ can be rewritten in terms of only two constants E and F as follows:‡

$$L_1^2 = \frac{1 - 4EF}{4E^4}, \quad L_2 = \frac{1 - 2EF}{2E^2}, \quad F = \frac{1}{2E}\left(1 - \frac{\eta_2}{c_1}\right). \tag{7.32}$$

Moreover, in view of (7.31) and the condition (7.27), it follows that the constants E and F must be restricted by the inequality

$$4EF < 1, \tag{7.33}$$

which also ensures that r is real.

The relation (7.28) between r and s, along with $(7.11)_{2,3}$, define a meridian curve on $\mathscr{s}$. Similarly, the relation between R and S for a corresponding meridian curve on $\mathscr{S}$ can be obtained directly from (7.28), $(7.20)_{2,3}$ and $(7.22)_2$ as

$$R^2 = (2/c_1)\,\{L_2 + L_1 \sin[(2c_1)^{\frac{1}{2}} ES + \alpha]\} = (1/\lambda_1^2)\,[L_2 + L_1 \sin(2\lambda_1 ES + \alpha)]. \tag{7.34}$$

With the use of (7.34), $(7.3)_2$, $(7.29)_3$ and $(7.32)_{1,2}$, it can now be verified that the requirements $(7.20)_{5,6}$ are identically satisfied.

‡ The constants E and F in (7.32) correspond to A and B of Green & Adkins (1970, ch. 4).

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Consider next the special case in which the surface load $f_3 = 0$ or equivalently when the parameter c_3 vanishes. In this case, (7.25) reduces to‡

$$\left[\frac{\eta_2}{c_1}R^2 - \frac{\eta_1}{2c_1}\right]^{-\frac{1}{2}} \mathrm{d}(R^2) = 2\,\mathrm{d}S. \tag{7.35}$$

Before integrating (7.35), it is expedient to define the minimum value of R, or equivalently the minimum value of r in view of $(7.20)_3$. Assuming that the arclength s is measured from the point at which r assumes its minimum value, we define the minimum value of r at $s = 0$ by

$$r_0 = r_{\min} = r(0). \tag{7.36}$$

Now in order for $(7.11)_1$ to have a minimum at $s = 0$, we must have $\mathrm{d}r/\mathrm{d}s = 0$ and $\mathrm{d}^2r/\mathrm{d}s^2 > 0$ at that point. Keeping this in mind, as well as the result that when $c_3 = 0$ the mean curvature $H = 0$ also by lemma 6.4, with the help of $(7.8)_2$, $(7.20)_{2,3}$ and (7.22) we integrate (7.35) to obtain

$$r^2 = s^2 + r_0^2, \quad R^2 = S^2 + r_0^2/\lambda_1^2, \tag{7.37}$$

$$\eta_1 = 4r_0^2 > 0, \quad \eta_2 = c_1 > 0. \tag{7.38}$$

The result $(7.37)_2$ identically satisfies both of the requirements $(7.20)_{5,6}$ with $c_3 = 0$. It should also be noted that $(7.37)_{1,2}$, regarded as solutions for $\mathscr{s}$ and $\mathscr{S}$, represent catenaries with parameters r_0 and r_0/λ_1 with their directrixes coinciding with the axis of revolution ($r = 0$) of the membrane.

The results of the present subsection (B) are obtained under the assumption that the deformed surface is also a complete surface of revolution which is restricted by (7.12) and this, in turn, requires the vanishing of the constant γ in the differential criterion (7.4). In this way, we have recovered all of the previously known results (see Green & Adkins 1970, ch. 4) for controllable solutions of a membrane of revolution with both of its initial and deformed states in the forms of complete surfaces of revolution.§ By contrast, in carrying out the development of subsection (A) only the initial surface was assumed to be in the form of a surface of revolution and this resulted in a deformed surface characterized by (7.5) and (7.6) including the case in which $\gamma \neq 0$ in (7.7). In this connection, it should be noted that the difference between the solution with $\gamma \neq 0$ and that for which $\gamma = 0$ is nontrivial in the following sense: the difference between the two categories of controllable solutions is not merely a deformation which could result from a rigid body displacement of the deformed surface in the first category into the deformed surface of the second category.

(C) *A membrane with its deformed configuration in the form of a surface of revolution*

We consider now a third class of elastic membranes whose deformed states are in the forms of surfaces of revolution but we do not invoke the assumption (7.12).‖ Thus, given any non-developable initial surface, we show that the meridian curves (7.28) and $(7.37)_1$ generate, in fact, all possible controllably deformed surfaces of revolution.

‡ The condition (7.35) may also be obtained directly from the criterion $(7.8)_1$ with $\gamma = 0$.

§ We have not recorded here the explicit expressions for ρ, f^3 and $N^{\alpha\beta}$ but these can be easily calculated from (7.28) and (7.34) when $f^3 \neq 0$ and from $(7.37)_{1,2}$ when $f^3 = 0$.

‖ Recall that the assumption (7.12) implies that a net of lines of curvature on $\mathscr{S}$ in the form of a sector of surface of revolution transforms into a net of lines of curvature on $\mathscr{s}$.

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Let the coordinate curves on $\mathscr{s}$ be specified by the surface coordinates y^α as defined by (7.13) and let the values of y^α be identified with those of the convected coordinates θ^α in the deformed configuration so that now

$$\theta^\alpha = y^\alpha = (s, \theta) \quad \text{on} \quad \mathscr{s}. \tag{7.39}$$

In view of (7.39) and recalling (7.13) and (7.18), it follows that $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the same as those given by $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$ in (7.18) with r and z defined by (7.11). The coefficients of the second fundamental form, of course, must be compatible with those in (5.21). Moreover, since by $(6.11)_2$ the mean curvature H must be a constant, with the help of $(2.17)_1$, (7.18) and $(7.11)_3$ we have

$$H = \frac{c_3}{c_1} = \pm \frac{[1-(rr^{\cdot})^{\cdot}]}{2r[1-(r^{\cdot})^2]^{\frac{1}{2}}}. \tag{7.40}$$

Also, by (5.7) and $(7.18)_1$, the coefficients of the first fundamental form of the initial surface are

$$A_{\alpha\beta} = \frac{2}{c_1}\begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix} \tag{7.41}$$

and we recall that both $A_{\alpha\beta}$ and $B_{\alpha\beta}$ must satisfy the differential criterion (5.22) and the initial compatibility requirements $(2.21)_{1,2}$.

Using (7.41), from a comparison of the expression for b_{12} as given by $(7.18)_2$ and $(5.21)_3$ we find

$$\bar{V} = 2n\pi \pm \tfrac{1}{2}\pi \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{7.42}$$

and this leads at once to

$$\bar{X}_\alpha = 0, \tag{7.43}$$

in view of (5.24). Hence the criterion (5.22) is automatically satisfied. From (5.23), (7.41), (6.12) and the dual of $(7.3)_2$, it follows that (7.43) with $\alpha = 1$ is identically satisfied while (7.43) with $\alpha = 2$ reduces to

$$\left\{\ln r^4\left[\left(\frac{c_3}{c_1}\right)^2 + \frac{r^{\cdot\cdot}}{r}\right]\right\}^{\cdot} = 0. \tag{7.44}$$

The first integral part of (7.44) yields

$$2c_1 r^{\cdot\cdot} + \left[c_3\left(\frac{2}{c_1}\right)^{\frac{1}{2}}\right]^2 r - \eta_1 \frac{c_1}{2} r^{-3} = 0, \tag{7.45}$$

where η_1 is an arbitrary constant of integration introduced earlier in (7.23).‡ Observing that (7.45) has the same form as (7.23), the integration of (7.45) can be carried out in a similar way. Thus, with the help of the dual of $(7.24)_1$, $(7.20)_{2,3}$ and (7.40), we can easily integrate the differential equation (7.45) in the form (7.28) for nonvanishing c_3 or in the form $(7.37)_1$ when $c_3 = 0$. With these results, as well as (7.41), (7.42), (6.12), the dual of $(7.3)_2$ and $(7.11)_3$, it is a simple matter to show that b_{11} and b_{22} as given by $(7.18)_2$ are indeed compatible with the corresponding expressions in (5.21).

We now turn to an examination of the compatibility equations of the initial surface. With the use of (6.12), (7.41), $(5.8)_2$, the expressions for $a_{\alpha\beta}$ as given by $(7.18)_1$ and the dual of $(7.3)_2$, as well as the derived expressions for r in the forms (7.28) and $(7.37)_1$, we deduce the following reduction of the initial compatibility equations $(2.21)_2$ and $(2.21)_1$:

(i) When $c_3 \neq 0$ (or equivalently $f^3 \neq 0$) to the set

$$B_{11}B_{22} - B_{12}^2 = \frac{2E^2}{c_1}\left[L_2 + L_1 \sin\beta + \frac{L_1^2 - L_2^2}{L_2 + L_1 \sin\beta}\right] \tag{7.46}$$

‡ Strictly speaking a different constant of integration, say η_4, should be used in (7.45). For later convenience, however, we use the same arbitrary constant η_1 as in (7.23).

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and
$$B_{11,2} = B_{12,1} + \frac{EL_1 \cos\beta}{L_2 + L_1 \sin\beta} B_{12},$$
$$B_{22,1} - (EL_1 \cos\beta)\, B_{11} - \frac{EL_1 \cos\beta}{L_2 + L_1 \sin\beta} B_{22} = B_{12,2}, \tag{7.47}$$
where
$$\beta = 2Es + \alpha \tag{7.48}$$
and E and α are defined by $(7.29)_{3,4}$.

(ii) When $c_3 = 0$ (or equivalently $f^3 = 0$) to the set
$$B_{11} B_{22} - B_{12}^2 = -\frac{2r_0^2}{c_1(s^2 + r_0^2)} \tag{7.49}$$
and
$$B_{11,2} = B_{12,1} + \frac{s}{s^2 + r_0^2} B_{12},$$
$$B_{22,1} - sB_{11} - \frac{s}{s^2 + r_0^2} B_{22} = B_{12,2}, \tag{7.50}$$
where r_0 is defined by (7.36).

The integration of (7.46)–(7.50) is, in general, quite difficult but it can be accomplished in special circumstances. We do not pursue the matter here any further, but note that if attention is restricted to such controllable deformations that the net of lines of curvature coordinates on $\mathscr{s}$ result from a corresponding set of lines of curvature coordinates on $\mathscr{S}$, i.e. if $A_{12} = B_{12} = 0$, then the differential equations (7.46)–(7.50) can be integrated in a straightforward manner although the process of integration is rather lengthy.

We close this section with a summary of the conclusions reached in the present subsection (C) by the following two corollaries to theorems 6.5–6.7:

Corollary 7.3. *In a controllable deformation by edge loads alone, the only nondevelopable surface of revolution in the deformed state is a catenary generated by* $(7.37)_1$ *provided that the first and the second fundamental forms of the initial surface are characterized by* (7.41) *and the solutions of* (7.49)–(7.50).

Corollary 7.4. *In a controllable deformation in which the loads include a uniform normal pressure* ($f^3 \neq 0$), *the only nondevelopable surface of revolution in the deformed state is that generated by the meridian curve* (7.28) *provided that the first and the second fundamental forms of the initial surface are characterized by* (7.41) *and the solutions of* (7.46)–(7.47).

8. Several families of solutions. An alternative description of controllable deformation

We consider in this section some special cases of the general results of § 6 and obtain alternative descriptions of the controllable solutions when each of the two surfaces $\mathscr{S}$ and $\mathscr{s}$ has a simple shape in the form of a plane, a right circular cylinder or a sphere. In particular, we discuss five families of controllable deformations for which the initial and the deformed surfaces are specified as follows:

Family 1. Both the initial surface $\mathscr{S}$ and the deformed surface $\mathscr{s}$ are planes.

Family 2. The initial surface $\mathscr{S}$ is a sector of a right circular cylinder of radius R (>0) and $\mathscr{s}$ is a plane.

Family 3. The initial surface $\mathscr{S}$ is a plane and $\mathscr{s}$ is a sector of a right circular cylinder of radius r (>0).

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Family 4. Both surfaces $\mathscr{S}$ and $\mathscr{s}$ are sectors of right circular cylinders. This includes the case of complete right circular cylinders, i.e. tubes.

Family 5. Both surfaces $\mathscr{S}$ and $\mathscr{s}$ are sectors of spheres (or complete spheres) of radii R and r, respectively.

The fact that both the initial and the deformed surfaces of the membrane are prescribed enables us to introduce suitable surface coordinates on the surfaces of the families 1–5 above, which surface coordinates collectively will be designated by Y^α on $\mathscr{S}$ and by y^α on $\mathscr{s}$. The deformation of $\mathscr{S}$ into $\mathscr{s}$ may then be described by a transformation of the form (7.14). Such an alternative description, as it will become evident presently, permits easy interpretation of our general results especially when each of the two surfaces $\mathscr{S}$ and $\mathscr{s}$ has a simple shape.

In what follows, both the initial and the deformed configurations of the membrane will be referred to a common reference frame which we take to be a set of fixed rectangular Cartesian axes. With reference to the initial and deformed surfaces of the five families of controllable deformation specified above, we also introduce the rectangular Cartesian, the cylindrical polar and the spherical polar coordinate systems in the initial configuration of the membrane and designate these by $Y^i = (Y^1, Y^2, Y^3) = (Y, Z, X)$, $Y^i = (\Theta, Z, R)$ and $Y^i = (\Theta, \Phi, R)$, respectively. Similarly, the corresponding coordinate systems in the deformed configuration will be denoted by $y^i = (y^1, y^2, y^3) = (y, z, x)$, $y^i = (\theta, z, r)$ and $y^i = (\theta, \phi, r)$, respectively. The common reference frame, as well as the initial and the deformed surfaces, to within a rigid body displacement of the deformed surface may be so located that the origin of this frame is on the same common axis of the initial and the deformed cylindrical surfaces, and is also at the same centre of the initial and the deformed spherical surfaces. Moreover, when the Z-axis is coincident with the axis of cylindrical surfaces, the (Y, Z) and the (y, z) coordinate planes coincide, respectively, with the plane surface in the initial and the deformed configurations of the membrane.

Let $\boldsymbol{E}_i = (\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{I})$ denote the unit base vectors along the (Y, Z, X)-coordinate axes. Then, the surface coordinates in the initial configuration, along with the position vector and the first and the second fundamental forms $\bar{A}_{\alpha\beta}$, $\bar{B}_{\alpha\beta}$ of $\mathscr{S}$ may be summarized as follows:

(1) When $\mathscr{S}$ is a plane,
$$Y^\alpha = (Y, Z), \quad \boldsymbol{R} = Y\boldsymbol{J} + Z\boldsymbol{K}, \tag{8.1}$$
$$\bar{A}_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \delta_{\alpha\beta}, \quad \bar{B}_{\alpha\beta} = 0. \tag{8.2}$$

(2) When $\mathscr{S}$ is a right circular cylinder,
$$Y^\alpha = (\Theta, Z), \quad \boldsymbol{R} = R\cos\Theta\boldsymbol{I} + R\sin\Theta\boldsymbol{J} + Z\boldsymbol{K}, \tag{8.3}$$
$$\bar{A}_{\alpha\beta} = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{B}_{\alpha\beta} = \begin{pmatrix} -R & 0 \\ 0 & 0 \end{pmatrix}. \tag{8.4}$$

(3) When $\mathscr{S}$ is a sphere,
$$Y^\alpha = (\Theta, \Phi), \quad \boldsymbol{R} = R\sin\Theta\cos\Phi\boldsymbol{I} + R\sin\Theta\sin\Phi\boldsymbol{J} + R\cos\Theta\boldsymbol{K}, \tag{8.5}$$
$$\bar{A}_{\alpha\beta} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\Theta \end{pmatrix}, \quad \bar{B}_{\alpha\beta} = \begin{pmatrix} -R & 0 \\ 0 & -R\sin^2\Theta \end{pmatrix}. \tag{8.6}$$

Similarly, with reference to the present deformed configuration, the surface coordinates, the position vector and the first and the second fundamental forms $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$ of $\mathscr{s}$ may be summarized as follows:

(1) When $\mathscr{s}$ is a plane,
$$y^\alpha = (y, z), \quad \boldsymbol{r} = y\boldsymbol{J} + z\boldsymbol{K}, \tag{8.7}$$
$$\bar{a}_{\alpha\beta} = \delta_{\alpha\beta}, \quad \bar{b}_{\alpha\beta} = 0. \tag{8.8}$$

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(2) When $\mathscr{s}$ is a right circular cylinder,

$$y^\alpha = (\theta, z), \quad \boldsymbol{r} = r\cos\theta \boldsymbol{I} + r\sin\theta \boldsymbol{J} + z\boldsymbol{K}, \tag{8.9}$$

$$\bar{a}_{\alpha\beta} = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix} = \text{const.}, \quad \bar{b}_{\alpha\beta} = \begin{pmatrix} -r & 0 \\ 0 & 0 \end{pmatrix} = \text{const.} \tag{8.10}$$

(3) When $\mathscr{s}$ is a sphere,

$$y^\alpha = (\theta, \phi), \quad \boldsymbol{r} = r\sin\theta\cos\phi \boldsymbol{I} + r\sin\theta\sin\phi \boldsymbol{J} + r\cos\theta \boldsymbol{K}, \tag{8.11}$$

$$\bar{a}_{\alpha\beta} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}, \quad \bar{b}_{\alpha\beta} = \begin{pmatrix} -r & 0 \\ 0 & -r\sin^2\theta \end{pmatrix}, \tag{8.12}$$

We now choose the values of the Y^α coordinates, i.e. the surface coordinates $(8.1)_1$, $(8.3)_1$ and $(8.5)_1$, of a typical point of $\mathscr{S}$ to be the convected coordinates at that point so that $(7.2)_1$ holds. Then, $\bar{A}_{\alpha\beta} = A_{\alpha\beta}$, $\bar{B}_{\alpha\beta} = B_{\alpha\beta}$; and, by virtue of theorems 6.5 and 6.6, the expressions for $a_{\alpha\beta}$, $b_{\alpha\beta}$ in the θ^α coordinates can be calculated from (5.17), (5.7), (5.26) and (5.27), using also $(8.2)_1$, $(8.4)_1$ and $(8.6)_1$. This leads to the results

$$a_{\alpha\beta} = \text{const.}, \quad b_{\alpha\beta} = \text{const.} \tag{8.13}$$

for any $\mathscr{s}$ in the families 1–4 and to the results

$$a_{\alpha\beta} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\Theta \end{pmatrix}, \quad b_{\alpha\beta} = \begin{pmatrix} \mp r & 0 \\ 0 & \mp r\sin^2\Theta \end{pmatrix}, \tag{8.14}$$

$$r = (\tfrac{1}{2}c_1)^{\frac{1}{2}} R$$

in the case of family 5. Next, we suppose that the deformation of the membrane is characterized by (7.14) or equivalently by (7.15) in view of $(7.2)_1$ and recall the transformation relations $(7.19)_{1,2}$ which must be satisfied by the functions $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$. Inasmuch as these functions are known explicitly, the transformation relations $(7.19)_{1,2}$ yield a system of partial differential equations for y^α in terms of θ^α. The solutions of these differential equations then determine the desired deformations.

If in the families 1–4 the surface $\mathscr{s}$ is a plane or a right circular cylinder, then it is easily seen from (8.8), (8.10), (8.13) and (7.15) that the constant values of $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$ in the y^γ-coordinates transform as constant values for $a_{\alpha\beta}$, $b_{\alpha\beta}$ in the θ^γ-coordinates. The corresponding Christoffel symbols calculated from $\bar{a}_{\alpha\beta}(y^\gamma)$ and $a_{\alpha\beta}(\theta^\gamma)$, respectively, are identically zero; and, from the well-known transformation law for the Christoffel symbols (not recorded here), we deduce the differential equations

$$\frac{\partial^2 y^\tau}{\partial\theta^\mu\,\partial\theta^\nu} = 0. \tag{8.15}$$

A solution of (8.15) is

$$y^\alpha = c^\alpha_\beta\,\theta^\beta + c^\alpha, \tag{8.16}$$

where c^α_β and c^α are integration constants. Having obtained (8.16) and remembering that both $\bar{b}_{\alpha\beta}$ and $b_{\alpha\beta}$ are constants, we can easily verify that $(7.19)_2$ yields no new information on the transformation (7.15). Replacing θ^β in (8.16) with Y^β in view of $(7.2)_1$ and observing, with the help of $(8.7)_1$ and $(8.9)_1$, that c^α correspond to rigid body displacements, we finally obtain the expression

$$y^\alpha = c^\alpha_\beta\,Y^\beta \quad (\det c^\alpha_\beta \neq 0), \tag{8.17}$$

which characterizes the controllable deformation for the families 1–4.

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In the case of family 5 in which s is a sphere, it follows from (8.12) and $(8.14)_{1,2}$ that both the y^1-curve and the θ^1-curve on s form the great circles of the same sphere. Hence, we may identify the y^1-curve and the θ^1-curve on s and measure the square of the element of arclength $ds_{(1)}$ along the y^1 (or θ^1) coordinate line by

$$ds^2_{(1)} = r^2(dy^1)^2 = r^2(d\theta^1)^2. \tag{8.18}$$

It follows from the above expressions that

$$y^1 = k_1 \pm \theta^1, \tag{8.19}$$

where k_1 is an integration constant. In view of this result, as well as (8.12) and (8.14), the solutions of the differential equations obtained from $(7.19)_{1,2}$ are

$$y^1 = n\pi \pm \theta^1, \quad y^2 = k_2 \pm \theta^2 \quad (n = 0, \pm 1, \ldots), \tag{8.20}$$

where k_2 is a constant of integration. Substituting $(7.2)_1$ into (8.20) and examining the resulting equations with the help of $(8.5)_1$ and $(8.11)_1$, we see that $k_2 = 0$ and, to within a rigid body rotation, $(8.20)_{1,2}$ reduce to

$$y^1 = Y^1 \text{ or } \pi - Y^1, \quad y^2 = Y^2. \tag{8.21}$$

We are now in a position to assign some interpretations to the deformations characterized by (8.17) and (8.21). For family 1, the coefficients c^1_1 and c^2_2 in (8.17) represent stretchings in the Y- and the Z-coordinate directions, while c^1_2 and c^2_1 represent shearings along the Y- and the Z-coordinate directions. For family 2, c^1_1 describes straightening about the Z-coordinate direction, c^2_2 characterizes extension along the Z-axis, while c^1_2 and c^2_1 represent shearing deformations along the Y- and the Z-coordinate directions, respectively. For family 3, c^1_1 and c^1_2 represent bending and torsion about the Z-axis, while c^2_1 and c^2_2 are measures of shearing and stretching along the Z-axis. For family 4, the physical meanings associated with c^1_2 and c^2_2 are the same as those of family 3. In addition, when $\mathscr{S}$ and s are sectors of cylinders in family 4, c^1_1 corresponds to straightening and/or bending about the Z-axis and c^2_1 represents shearing along the Z-coordinate direction; but, when $\mathscr{S}$ and s are tubes in family 4, then c^α_β must be further restricted by

$$c^1_1 = \pm 1, \quad c^2_1 = 0 \quad (c^2_2 \neq 0), \tag{8.22}$$

which includes the case of a tube turned inside out and subjected to no shear. The change of radius from R to r in family 4 represents the amount of expansion of cylindrical surface $\mathscr{S}$, while the deformation (8.21) along with $(8.14)_3$ characterize the expansion or inversion of spheres in family 5.

It is convenient to summarize the results obtained so far in this section by the following two lemmas:

Lemma 8.1. *Suppose that each of the two initial surfaces $\mathscr{S}$ and the deformed surface s is either a plane or a right circular cylinder and suppose that the surface coordinates on $\mathscr{S}$ and s are specified by* $(8.1)_1$, $(8.3)_1$, $(8.7)_1$ *and* $(8.9)_1$. *Then, the general results of theorem* 6.5 *given by the solutions* (5.17) *and* (5.27), *or alternatively* $(8.13)_{1,2}$ *after using* $(7.2)_1$, *are equivalent to the deformation* (8.17).

Lemma 8.2. *Suppose that both $\mathscr{S}$ and s are spheres and that the surface coordinates of these surfaces are specified by* $(8.5)_1$ *and* $(8.11)_1$, *respectively. Then, the general results of theorem* 6.6 *given by the solution* (5.7) *and* (5.26), *or alternatively* $(8.14)_{1,2}$ *after using* $(7.2)_1$, *are equivalent to the deformation* (8.21).

To complete the present development, we need to calculate the expressions for the mass density ρ, the strain invariants, the components $N^{\alpha\beta}$ and the uniform normal pressure f^3 in terms of

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the alternative kinematic descriptions for each of the five families of controllable deformations listed earlier in this section. Since the calculations are straightforward, we omit details and record only the final expressions. Thus, remembering also the above interpretations which can be associated with the coefficients c^α_β in (8.17), we may summarize the results as follows:

Family 1 – *Stretching and shearing of a plane* $\mathscr{S}$ *into another plane* $\mathfrak{s}$.‡ The initial and the deformed membranes in this family are specified by $(8.1)_2$ and (4.1) and by $(8.7)_2$ and

$$\rho = \rho_0 \,|\det c^\alpha_\beta|^{-1}, \tag{8.23}$$

respectively. The deformation is described by (8.17) with the surface coordinates y^α and Y^α specified by $(8.7)_1$ and $(8.1)_1$ respectively. In view of the particular choice of the convected coordinates specified by $(7.2)_1$ and $(8.1)_1$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ of the initial surface are given by (8.2), while $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the deformed surface are calculated to be

$$a_{\alpha\beta} = \delta_{\mu\nu}\, c^\mu_\alpha c^\nu_\beta \equiv c^\gamma_\alpha c^\gamma_\beta, \quad b_{\alpha\beta} = 0. \tag{8.24}$$

The corresponding expressions for $N^{\alpha\beta}$ and f^3 are

$$N^{\alpha\beta} = 2\rho(\Psi_1 \delta^{\alpha\beta} + 2\Psi_2 c^\gamma_\alpha c^\gamma_\beta), \quad f^3 = 0, \tag{8.25}$$

where the arguments I_1, I_2 of the functions Ψ_γ are calculated to be

$$I_1 = \delta^{\alpha\beta}\delta_{\xi\eta}\, c^\xi_\alpha c^\eta_\beta \equiv c^\gamma_\alpha c^\gamma_\alpha, \tag{8.26}$$
$$I_2 = \delta^{\alpha\xi}\delta^{\beta\eta}\delta_{\mu\nu}\delta_{\sigma\tau}\, c^\mu_\alpha c^\nu_\beta c^\sigma_\xi c^\tau_\eta \equiv c^\mu_\alpha c^\mu_\beta c^\nu_\alpha c^\nu_\beta.$$

Family 2 – *Stretching, shearing and straightening of a sector of a right circular cylinder of radius* R (>0) *into a plane*. The initial and the deformed membranes in this family are specified by $(8.3)_2$ and (4.1) and by $(8.7)_2$ and

$$\rho = \rho_0 R\,|\det c^\alpha_\beta|^{-1}, \tag{8.27}$$

respectively. The deformation is described by (8.17), together with $(8.7)_1$ and $(8.3)_1$. In view of the particular choice of the convected coordinates specified by $(7.2)_1$ and $(8.3)_1$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ of the initial surface are given by (8.4), while $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the deformed surface are found to be the same as (8.24). The corresponding expressions for $N^{\alpha\beta}$ and f^3 are

$$\left.\begin{aligned} N^{11} &= \frac{2\rho}{R^2}\left(\Psi_1 + 2\frac{c^\gamma_1 c^\gamma_1}{R^2}\Psi_2\right), \quad N^{22} = 2\rho(\Psi_1 + 2c^\gamma_2 c^\gamma_2 \Psi_2), \\ N^{12} &= N^{21} = \frac{4\rho}{R^2} c^\gamma_1 c^\gamma_2 \Psi_2, \quad f^3 = 0, \end{aligned}\right\} \tag{8.28}$$

where the strain invariants which occur in the arguments of the functions Ψ_γ are

$$I_1 = \frac{c^\gamma_1 c^\gamma_1}{R^2} + c^\gamma_2 c^\gamma_2, \quad I_2 = \left(\frac{c^\gamma_1 c^\gamma_1}{R^2}\right)^2 + 2\left(\frac{c^\gamma_1 c^\gamma_2}{R}\right)^2 + (c^\gamma_2 c^\gamma_2)^2. \tag{8.29}$$

Family 3 – *Stretching, shearing, bending and torsion of a plane into a sector of a right circular cylinder of radius* r (>0). The initial and the deformed membranes in this family are specified by $(8.1)_2$ and (4.1) and by $(8.9)_2$ and

$$\rho = \rho_0 r^{-1} |\det c^\alpha_\beta|^{-1}, \tag{8.30}$$

respectively. The deformation is described by (8.17), together with $(8.9)_1$ and $(8.1)_1$. In view of the choice of the convected coordinates specified by $(7.2)_1$ and $(8.1)_1$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ of the initial surface are given by (8.2), while $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the deformed surface are calculated to be

$$a_{\alpha\beta} = r^2 c^1_\alpha c^1_\beta + c^2_\alpha c^2_\beta, \quad b_{\alpha\beta} = -\mu r c^1_\alpha c^1_\beta, \quad \mu = \frac{|\det c^\alpha_\beta|}{\det c^\alpha_\beta} = \pm 1. \tag{8.31}$$

‡ In the rest of this section, for convenience we deviate from the usual summation convention employed in the previous parts of the paper and sum over repeated indices on the same level as in $(8.24)_1$.

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The corresponding components of $N^{\alpha\beta}$ and the uniform normal pressure f^3 are given by

$$\left.\begin{aligned} N^{\alpha\beta} &= 2\rho[\Psi_1\delta^{\alpha\beta}+2(r^2c^1_\alpha c^1_\beta+c^2_\alpha c^2_\beta)\,\Psi_2],\\ f^3 &= 2\mu r\{c^1_\alpha c^1_\alpha\Psi_1+2[(rc^1_\alpha c^1_\alpha)^2+(c^1_\alpha c^2_\alpha)^2]\,\Psi_2\},\end{aligned}\right\} \tag{8.32}$$

where the arguments I_1, I_2 of the functions Ψ_r $(r = 1, 2)$ are given by

$$I_1 = r^2c^1_\alpha c^1_\alpha+c^2_\alpha c^2_\alpha, \quad I_2 = (r^2c^1_\alpha c^1_\beta+c^2_\alpha c^2_\beta)\,(r^2c^1_\alpha c^1_\beta+c^2_\alpha c^2_\beta). \tag{8.33}$$

In view of the forms of the strain invariants (8.33) the strain energy density in (2.29) can be regarded as a different function $\check{\psi}$ of r and c^α_β, i.e.

$$\psi = \hat{\psi}(I_1, I_2) = \check{\psi}(r, c^\alpha_\beta). \tag{8.34}$$

Then, by use of (8.33) and the chain rule for differentiation, $(8.32)_2$ can alternatively be represented as

$$f^3 = \mu\frac{\partial\check{\psi}}{\partial r}. \tag{8.35}$$

Family 4 – Expansion, straightening and/or bending, stretching, shearing and torsion of a sector of a right circular cylinder of radius R (> 0) into a sector of another right circular cylinder of radius r (> 0), as well as expansion, turning inside out, stretching and torsion of a tube (i.e. a complete circular cylinder) into another tube. Consider first the more general case of a right circular cylinder. The initial and the deformed membranes in this family are specified by $(8.3)_2$ and (4.1) and by $(8.9)_2$ and

$$\rho = \rho_0\frac{R}{r}|\det c^\alpha_\beta|^{-1}, \tag{8.36}$$

respectively. In view of the choice of the convected coordinates specified by $(7.2)_1$ and $(8.3)_1$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ of the initial surface $\mathscr{S}$ are given by (8.4), while $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the deformed surface $\mathscr{s}$ are found to be the same as (8.31). The corresponding expressions for $N^{\alpha\beta}$ and f^3 are

$$\left.\begin{aligned} N^{11} &= \frac{2\rho}{R^2}\left[\Psi_1+2\,\frac{(rc^1_1)^2+(c^2_1)^2}{R^2}\,\Psi_2\right],\\ N^{22} &= 2\rho\{\Psi_1+2[(rc^1_2)^2+(c^2_2)^2]\,\Psi_2\},\\ N^{12} &= N^{21} = \frac{4\rho}{R^2}(r^2c^1_1c^1_2+c^2_1c^2_2)\,\Psi_2,\\ f^3 &= 2\mu r\left[\left(\frac{c^1_1}{R}\right)^2+(c^1_2)^2\right]\Psi_1+4\mu r\left[\frac{r^2(c^1_1)^4+(c^1_1c^2_1)^2}{R^4}\right.\\ &\qquad\left.+2\,\frac{(rc^1_1c^1_2)^2+c^1_1c^1_2c^2_1c^2_2}{R^2}+r^2(c^1_2)^4+(c^1_2c^2_2)^2\right]\Psi_2\\ &= \mu\frac{\partial\check{\psi}}{\partial r}.\end{aligned}\right\} \tag{8.37}$$

In (8.37), the functions Ψ_r $(r = 1, 2)$ depend on the strain invariants

$$\left.\begin{aligned} I_1 &= \frac{(rc^1_1)^2+(c^2_1)^2}{R^2}+(rc^1_2)^2+(c^2_2)^2,\\ I_2 &= \frac{(rc^1_1)^2+(c^2_1)^2}{R^4}+\frac{2(r^2c^1_1c^1_2+c^2_1c^2_2)^2}{R^2}+[(rc^1_2)^2+(c^2_2)^2]^2\end{aligned}\right\} \tag{8.38}$$

and, in view of (8.38), the strain energy density in (2.29) has been expressed as a different function $\check{\psi}$ or r, R and c^α_β, i.e.

$$\psi = \hat{\psi}(I_1, I_2) = \check{\psi}(r, R, c^\alpha_\beta). \tag{8.39}$$

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The foregoing results are valid when $\mathscr{S}$ is a sector of a right circular cylinder, including the special case of a tube. In the latter case the results simplify considerably and corresponding to the expressions in (8.31), (8.36), (8.37), (8.38) and (8.39) we have in the order listed

$$\left.\begin{gathered} a_{\alpha\beta} = \begin{pmatrix} r^2 & \pm r^2 c_2^1 \\ \pm r^2 c_2^1 & (rc_2^1)^2 + (c_2^2)^2 \end{pmatrix}, \quad b_{\alpha\beta} = -\mu r \begin{pmatrix} 1 & \pm c_2^1 \\ \pm c_2^1 & (c_2^1)^2 \end{pmatrix}, \\ \mu = \frac{|c_2^2|}{c_2^2} = \pm 1, \quad c_2^2 \neq 0, \end{gathered}\right\} \tag{8.40}$$

$$\rho = \rho_0 \frac{R}{r|c_2^2|}, \tag{8.41}$$

$$\left.\begin{gathered} N^{11} = (2\rho/R^2)\,[\Psi_1 + 2(r/R)^2\,\Psi_2], \quad N^{22} = 2\rho\{\Psi_1 + 2[(rc_2^1)^2 + (c_2^2)^2]\,\Psi_2\}, \\ N^{12} = N^{21} = (4\rho/R^2)\,(\pm 1)\,r^2 c_2^1 \Psi_2, \\ f^3 = 2\mu r \left\{\left[\frac{1}{R^2} + (c_2^1)^2\right]\Psi_1 + 2\left[\frac{r^2}{R^4} + 2\left(\frac{rc_2^1}{R}\right)^2 + r^2(c_2^1)^4 + (c_2^1 c_2^2)^2\right]\Psi_2\right\} = \mu\frac{\partial\acute{\psi}}{\partial r}, \end{gathered}\right\} \tag{8.42}$$

$$\left.\begin{gathered} I_1 = (r/R)^2 + (rc_2^1)^2 + (c_2^2)^2, \\ I_2 = r^4/R^4 + 2(r^2 c_2^1/R)^2 + [(rc_2^1)^2 + (c_2^2)^2]^2 \end{gathered}\right\} \tag{8.43}$$

and

$$\psi = \hat{\psi}(I_1, I_2) = \acute{\psi}(r, R, c_2^1, c_2^2). \tag{8.44}$$

Family 5 – Expansion or eversion of a sector of a sphere of radius R into a sector of another sphere of radius r. The initial and the deformed membranes in this family are specified by $(8.5)_2$ and (4.1) and by $(8.11)_2$ and

$$\rho = \rho_0 (R/r)^2, \tag{8.45}$$

respectively. In view of the choice of convected coordinates specified by $(7.2)_1$ and $(8.5)_1$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ of the initial surface $\mathscr{S}$ are given by (8.6), while $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the deformed surface $\mathscr{s}$ are given by (8.14). The corresponding expressions for $N^{\alpha\beta}$ and f^3 are

$$\left.\begin{gathered} N^{11} = N^{22}\sin^2\Theta = \frac{2\rho}{R^2}\left(\Psi_1 + 2\frac{r^2}{R^2}\Psi_2\right) = \frac{\rho}{2r}\frac{\partial\acute{\psi}}{\partial r}, \\ N^{12} = N^{21} = 0, \\ f^3 = \pm 4\frac{r}{R^2}\left[\Psi_1 + 2\left(\frac{r}{R}\right)^2\Psi_2\right] = \pm\frac{\partial\acute{\psi}}{\partial r}, \end{gathered}\right. \tag{8.46}$$

where the strain invariants which occur in the arguments of Ψ_γ are

$$I_1 = 2\left(\frac{r}{R}\right)^2, \quad I_2 = 2\left(\frac{r}{R}\right)^4 \tag{8.47}$$

and, in view of (8.47), the strain energy density can be expressed in terms of a different function of r and R as

$$\psi = \hat{\psi}(I_1, I_2) = \acute{\psi}(r, R). \tag{8.48}$$

Recalling lemmas 8.1 and 8.2, the foregoing results may be summarized by the following two theorems:

THEOREM 8.1. *If the initial surface is either a plane or a right circular cylinder and if the normal pressure $f^3 = 0$, then the solutions for families 1 and 2 given above are the only possible controllable solutions in every isotropic elastic membrane whose strain energy density is given by (2.29) and whose initial mass density is uniform throughout $\mathscr{S}$.*

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THEOREM 8.2. *If the initial surface is a plane, a sector of a right circular cylinder or a sector of a sphere, then under a nonzero uniform normal pressure the solutions for families* 3–5 *above are the only possible controllable solutions in every isotropic elastic membrane whose strain energy density is given by* (2.29) *and whose initial mass density is uniform throughout* $\mathscr{S}$.

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APPENDIX A

This appendix provides details of the derivation of the solution of the system of differential equations (5.10) when $\bar{K} = 0$.

First, with the help of (2.13), (2.14) and the dual of $(2.8)_2$, we note that it is easy to show the partial differential equations (5.10) imply $(4.8)_1$ and (4.12). The last two algebraic equations with the help of the dual of $(2.4)_2$ can be readily expanded and solved for $a_{11}A_{22}$, $a_{22}A_{11}$ in the forms

$$\left.\begin{aligned} a_{11}A_{22} &= a_{12}A_{12} + \tfrac{1}{2}[c_1 A \pm (AA_{11}A_{22}U)^{\frac{1}{2}}],\\ a_{22}A_{11} &= a_{12}A_{12} + \tfrac{1}{2}[c_1 A \mp (AA_{11}A_{22}U)^{\frac{1}{2}}],\end{aligned}\right\} \tag{A 1}$$

where U is defined by
$$U = (2c_2 - c_1^2) - u^2 \geqq 0, \quad u = \frac{2a_{12} - c_1 A_{12}}{(A_{11}A_{22})^{\frac{1}{2}}} \tag{A 2}$$

and the nonnegative restriction imposed on U is demanded by the fact that each of $a_{11}A_{22}$, $a_{22}A_{11}$ is real.

In what follows, we consider separately the two cases in which U assumes zero or positive values everywhere on the surface $\mathscr{s}$. In the special case that U vanishes everywhere,‡ from (A 2), (A 1) and $(4.11)_2$ we readily obtain

$$\left.\begin{aligned} \frac{a_{11}}{A_{11}} = \frac{a_{22}}{A_{22}} &= \tfrac{1}{2}c_1\left[1 \pm \left(\frac{2c_2}{c_1^2} - 1\right)^{\frac{1}{2}}(A_{11}A_{22})^{-\frac{1}{2}}A_{12}\right],\\ a_{12} &= \tfrac{1}{2}c_1\left[A_{12} \pm \left(\frac{2c_2}{c_1^2} - 1\right)^{\frac{1}{2}}(A_{11}A_{22})^{\frac{1}{2}}\right].\end{aligned}\right\} \tag{A 3}$$

‡ The case in which U may vanish at some isolated points will be excluded. In fact, henceforth we exclude consideration of isolated points at which the value of a function such as U is zero, especially when differentiation of the function is required.

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In the case when U is positive, using $(2.10)_{4,5}$ and the dual of $(2.4)_2$, expansion of (5.10) with indices $(\alpha, \beta, \gamma) = (1, 2, 1)$ and $(1, 2, 2)$ leads to the differential equations

$$\left.\begin{aligned} 2a_{12,1} - a_{12}\frac{A_{,1}}{A} &= \frac{1}{A}\left\{(a_{11}A_{22})\left[A_{11,2} - A_{12}\frac{A_{22,1}}{A_{22}}\right] + (a_{22}A_{11})\left[2A_{12,1} - A_{11,2} - A_{12}\frac{A_{11,1}}{A_{11}}\right]\right\}, \\ 2a_{12,2} - a_{12}\frac{A_{,2}}{A} &= \frac{1}{A}\left\{(a_{11}A_{22})\left[2A_{12,2} - A_{22,1} - A_{12}\frac{A_{22,2}}{A_{22}}\right] + (a_{22}A_{11})\left[A_{22,1} - A_{12}\frac{A_{11,2}}{A_{11}}\right]\right\}. \end{aligned}\right\} \tag{A 4}$$

After substituting (A 1) into (A 4), dividing both sides of the resulting equations by (the nonzero) $U^{\frac{1}{2}}$ and simplifying we obtain

$$-\frac{u_{,\alpha}}{\pm[(2c_2 - c_1^2) - u^2]^{\frac{1}{2}}} = X_\alpha, \tag{A 5}$$

where the expressions for X_α on the right-hand side of (A 5) are defined by (5.19). It is important to note that the main reason for the reduction of (A 4) to the forms (A 5) is the fact that X_α on the right-hand side of (A 5) involve only the metric coefficients $A_{\alpha\beta}$ on the initial surface.

If $2c_2 - c_1^2 \neq 0$, then it must be positive according to $(4.11)_2$. Hence, the function

$$v = \arccos\left[\frac{u}{\pm(2c_2 - c_1^2)^{\frac{1}{2}}}\right] \quad (2c_2 - c_1^2 > 0) \tag{A 6}$$

is well defined and the differential equations (A 5) may be rewritten as

$$v_{,\alpha} = X_\alpha. \tag{A 7}$$

Further, by using (5.19), the expanded version of (2.11) and $(2.21)_2$, it can be seen that the only nontrivial integrability condition for (A 7), namely

$$X_{1,2} - X_{2,1} = 2A^{\frac{1}{2}}\bar{K} = 0, \tag{A 8}$$

is identically satisfied as to be expected. Therefore, (A 7) can be integrated to yield

$$v = V, \tag{A 9}$$

where V is given by (5.18). Next, by using (A 6) with its left-hand side now specified by the solution (A 9), from $(A\,2)_2$ and $(A\,1)_{1,2}$ follow the solutions (5.17).

The last results have been obtained under the condition $(A\,6)_2$. Consider now the case in which $2c_2 - c_1^2 = 0$. Then, by applying the argument of real numbers directly to (4.10) and using $(4.8)_1$, we obtain immediately the result (5.7). Evidently, the expressions (A 3) for vanishing values of U, as well as (5.7) for vanishing values of $(2c_2 - c_1^2)$, are contained in the more general results (5.17). Moreover, it can be easily verified that (5.17) does indeed satisfy the differential equations (5.10) identically. Hence, (5.17) is the desired solution for $a_{\alpha\beta}$ when $\bar{K} = 0$.

Appendix B

This appendix provides details of the derivation of the solutions for $b_{\alpha\beta}$ from $(4.8)_5$, (2.18) and (2.19).

In view of the results (5.17) and (5.7), as well as $(5.8)_2$ which is equivalent to (5.7) or (5.17), the expression for the component R_{1212} of the curvature tensor in the deformed configuration can be calculated (in terms of the Christoffel symbols) from $(2.6)_{4,5}$ and $(2.7)_2$; and this result, with the help of $(2.10)_5$ and the duals of $(2.6)_{4,5}$, can be related to $\bar{R}_{1212}$ in the initial undeformed configuration. The resulting expression after using also the dual of (2.20) and $(2.21)_2$ becomes

$$R_{1212} = \tfrac{1}{2}c_1 A\bar{K}. \tag{B 1}$$

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With the use of the last relation, the two algebraic equations (2.19) and $(4.8)_5$ can be solved for $b_{11}A_{22}$, $b_{22}A_{11}$ in the forms

$$\begin{aligned} b_{11}A_{22} &= b_{12}A_{12} + \tfrac{1}{2}[c_3 A \pm (AA_{11}A_{22}\bar{U})^{\frac{1}{2}}], \\ b_{22}A_{11} &= b_{12}A_{12} + \tfrac{1}{2}[c_3 A \mp (AA_{11}A_{22}\bar{U})^{\frac{1}{2}}], \end{aligned} \tag{B 2}$$

where $\bar{U}$ is defined by
$$\bar{U} = (c_3^2 - 2c_1\bar{K}) - \bar{u}^2 \geqslant 0, \quad \bar{u} = \frac{2b_{12} - c_3 A_{12}}{(A_{11}A_{22})^{\frac{1}{2}}}, \tag{B 3}$$

and the nonnegative restriction imposed on $\bar{U}$ is demanded by the fact that each of $b_{11}A_{22}$, $b_{22}A_{11}$ is real. Next, with the use of $(5.8)_2$, from the expanded version of the Mainardi–Codazzi relations (2.18) we obtain the two differential equations

$$\left.\begin{aligned} &b_{12,1} + \frac{1}{2A}\left[A_{11}A_{22}\left(-\frac{A_{11,1}}{A_{11}} + \frac{A_{22,1}}{A_{22}}\right) + 2A_{12}(A_{12,1} - A_{11,2})\right] b_{12} \\ &\qquad = b_{11,2} + \frac{1}{2A}\left[(b_{11}A_{22})\left(-A_{11,2} + A_{12}\frac{A_{22,1}}{A_{22}}\right)\right. \\ &\qquad\qquad \left. + (b_{22}A_{11})\left(-A_{11,2} - A_{12}\frac{A_{11,1}}{A_{11}} + 2A_{12,1}\right)\right], \\ &b_{12,2} + \frac{1}{2A}\left[A_{11}A_{22}\left(\frac{A_{11,2}}{A_{11}} - \frac{A_{22,2}}{A_{22}}\right) + 2A_{12}(A_{12,2} - A_{22,1})\right] b_{12} \\ &\qquad = b_{22,1} + \frac{1}{2A}\left[(b_{11}A_{22})\left(-A_{22,1} - A_{12}\frac{A_{22,2}}{A_{22}} + 2A_{12,2}\right)\right. \\ &\qquad\qquad \left. + (b_{22}A_{11})\left(-A_{22,1} + A_{12}\frac{A_{11,2}}{A_{11}}\right)\right]. \end{aligned}\right\} \tag{B 4}$$

Since the analysis that follows requires differentiation of b_{11} and b_{22}, we consider separately the two cases in which $\bar{U}$ assumes zero or positive values everywhere on $\mathscr{s}$. In the special case that $\bar{U}$ vanishes, it follows from (B 2) and (B 3) that

$$\left.\begin{aligned} \frac{b_{11}}{A_{11}} = \frac{b_{22}}{A_{22}} &= \tfrac{1}{2}[c_3 \pm (c_3^2 - 2c_1\bar{K})^{\frac{1}{2}}(A_{11}A_{22})^{-\frac{1}{2}}A_{12}], \\ b_{12} &= \tfrac{1}{2}[c_3 A_{12} \pm (c_3^2 - 2c_1\bar{K})^{\frac{1}{2}}(A_{11}A_{22})^{\frac{1}{2}}], \end{aligned}\right\} \tag{B 5}$$

provided $\bar{K} \leqslant c_3^2/2c_1$.

Before continuing our general development for obtaining the necessary restrictions on $A_{\alpha\beta}$ and $\bar{K}$ from (B 4), we observe that in the special case in which $\bar{K} = c_3^2/2c_1$ the expressions (B 5) can be specialized at once to the forms (5.26) and these satisfy $(\text{B }4)_{1,2}$ identically. In the more general situation where $\bar{K} < c_3^2/2c_1$, after introducing (B 5) into $(\text{B }4)_{1,2}$, the restrictions on $A_{\alpha\beta}$ and $\bar{K}$ can be expressed as

$$\bar{X}_\alpha = 0, \tag{B 6}$$

where $\bar{X}_\alpha$ are defined by $(5.23)_{1,2}$.

In the case in which $\bar{U}$ is positive everywhere on $\mathscr{s}$, we may use (B 2) to eliminate $b_{11}A_{22}$, $b_{22}A_{11}$, $b_{11,2}$ and $b_{22,1}$ in (B 4) and then for simplicity replace the variable b_{12} in the resulting expressions by $\bar{u}$ in $(\text{B }3)_2$ to obtain

$$\left.\begin{aligned} &\bar{u}_{,1} + \bar{u}_{,2}\left[\frac{1}{A_{22}}\left(-A_{12} \pm \frac{\bar{u}A^{\frac{1}{2}}}{\bar{U}^{\frac{1}{2}}}\right)\right] \\ &= \bar{u}\left\{\frac{1}{A_{22}}\left[-A_{22,1} + A_{12,2} + \tfrac{1}{2}A_{12}\left(\frac{A_{11,2}}{A_{11}} - \frac{A_{22,2}}{A_{22}}\right)\right]\right\} \\ &\quad \pm \bar{U}^{\frac{1}{2}}\left\{\frac{(AA_{11}A_{22})^{\frac{1}{2}}{}_{,2}}{A_{22}(A_{11}A_{22})^{\frac{1}{2}}} - \frac{A_{22,2}A^{\frac{1}{2}}}{A_{22}^2} + \tfrac{1}{2}A^{-\frac{1}{2}}\left[A_{12}\frac{(A_{11}A_{22})_{,1}}{A_{11}A_{22}} - 2A_{12,1}\right]\right\} \mp c_1\frac{A^{\frac{1}{2}}\bar{K}_{,2}}{A_{22}\bar{U}^{\frac{1}{2}}} \end{aligned}\right\} \tag{B 7}$$

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and
$$\left.\begin{aligned}&\bar{u}_{,1}\left[\frac{1}{A_{11}}\left(-A_{12}\mp\frac{\bar{u}A^{\frac{1}{2}}}{\bar{U}^{\frac{1}{2}}}\right)\right]+\bar{u}_{,2}\\&=\bar{u}\left\{\frac{1}{A_{11}}\left[-A_{11,2}+A_{12,1}+\tfrac{1}{2}A_{12}\left(-\frac{A_{11,1}}{A_{11}}+\frac{A_{22,1}}{A_{22}}\right)\right]\right\}\\&\mp\bar{U}^{\frac{1}{2}}\left\{\frac{(AA_{11}A_{22})^{\frac{1}{2}}{}_{,1}}{A_{11}(A_{11}A_{22})^{\frac{1}{2}}}-\frac{A_{11,1}A^{\frac{1}{2}}}{A_{11}^{2}}+\tfrac{1}{2}A^{-\frac{1}{2}}\left[A_{12}\frac{(A_{11}A_{22})_{,2}}{A_{11}A_{22}}-2A_{12,2}\right]\right\}\pm c_1\frac{A^{\frac{1}{2}}\bar{K}_{,1}}{A_{11}\bar{U}^{\frac{1}{2}}}.\end{aligned}\right\}\quad\text{(B 8)}$$

From $(\mathrm{B}\,3)_1$ and the restriction $\bar{U}>0$ we have $\bar{K}<c_3^2/2c_1$; and it follows that the determinant of the coefficients of $\bar{u}_{,\alpha}$ in (B 7) and (B 8), namely $A(c_3^2-2c_1\bar{K})/(A_{11}A_{22}\bar{U})$, does not vanish. Keeping this and $(\mathrm{B}\,3)_1$ in mind, from $(\mathrm{B}\,7)_{1,2}$ we can solve for $\bar{u}_{,\alpha}$ in the form

$$\bar{u}_{,\alpha}=\bar{u}\frac{(c_3^2-2c_1\bar{K})_{,\alpha}}{2(c_3^2-2c_1\bar{K})}-(\pm\bar{X}_\alpha)\,\bar{U}^{\frac{1}{2}},\quad\text{(B 9)}$$

where $\bar{X}_\alpha$ are given by (5.23). Dividing both sides of (B 9) by (the nonzero) $\bar{U}^{\frac{1}{2}}$, making use of $(\mathrm{B}\,3)_1$ and introducing the well-defined function $\bar{v}$ by

$$\bar{v}=\arccos\,[\bar{u}/\pm(c_3^2-2c_1\bar{K})^{\frac{1}{2}}],\quad\text{(B 10)}$$

we finally obtain
$$\bar{v}_{,\alpha}=\bar{X}_\alpha.\quad\text{(B 11)}$$

In order to ensure the existence of the function $\bar{v}$ (or equivalently b_{12}, in view of (B 10) and $(\mathrm{B}\,3)_2$), we must require $\bar{K}$ to be of class C^2 and that $\bar{X}_\alpha$ be restricted by the integrability condition (5.22), i.e. we must find solutions of (B 11) such that $A_{\alpha\beta}$ and $\bar{K}$ satisfy the criterion (5.22). Once (5.22) is satisfied, (B 11) can be integrated to yield

$$\bar{v}=\bar{V},\quad\text{(B 12)}$$

where $\bar{V}$ is given by (5.24). Next, in view of (B 10) with its left-hand side now specified by the solution (B 12), from (B 3) and $(\mathrm{B}\,2)_{1,2}$ follow the solutions (5.21).

With the help of $(\mathrm{B}\,3)_{1,2}$, it can be easily shown that by setting $\bar{V}=0$ and assuming $\bar{K}<c_3^2/2c_1$, the criterion (B 6) and the corresponding result in (B 5) can be readily recovered from (5.22) and (5.21), respectively. Hence, the above solutions of $(4.8)_5$, (2.18) and (2.19) for $b_{\alpha\beta}$ may be summarized as follows:

(i) When $\bar{K}<c_3^2/2c_1$ and the criterion (5.22) is satisfied, the solution for $b_{\alpha\beta}$ is given by (5.21);
(ii) when $\bar{K}=c_3^2/2c_1\geqslant 0$, the solution for $b_{\alpha\beta}$ is given by (5.26); and
(iii) there are no solutions when $\bar{K}>c_3^2/2c_1$.